Composing Navigation Functions on Cartesian Products of Manifolds with Boundary

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Abstract. Given two compact, simply connected manifolds with boundary, and a navigation function (NF) on each manifold, this paper presents a simple composition law that yields a new NF on the cross product space. The method provides tunable "hooks" for shaping the new potential function while still guaranteeing obstacle avoidance and essentially global convergence. The composition law is associative, and successive compositions fold into a single, computational simple expression, enabling the practical construction of NFs on the Cartesian product of several manifolds.

1 Introduction

The gradient vector field of a properly designed artificial potential function can steer a robot to a goal, while avoiding obstacles along the way. Adding a damping term to flush out any unwanted kinetic energy generalizes this approach to the second-order setting, since total energy always decreases in damped mechanical systems [14]. Of course, if a well-tuned robot control system is already in place, the potential field may be used in a more traditional first-order manner to generate trajectories via the gradient flow (in the case of a position controller) or field itself (in the case of a velocity controller).

The problem of spurious minima and safety for second-order plants lead Rimon and Koditschek to introduce navigation functions (NFs), a refined notion of artificial potential functions [16]. They showed that every smooth compact connected manifold with boundary, $\mathcal{M}$, admits a smooth NF, $\varphi : \mathcal{M} \to \mathbb{R}$. Thus, given a fully actuated Lagrangian system evolving on a such a configuration manifold, the machinery of NFs "solves" the global dynamical control and obstacle avoidance problem [20]. *Caveat emptor:* constructing NFs for arbitrary manifolds remains an art: each new model space requires a handcrafted NF. Example constructions for special model spaces include "sphere" worlds [16] and "star" worlds [19].

While a general methodology for constructing NFs seems unlikely, an "NF designers toolbox" for configuration spaces of practical interest seems plausible. Cartesian product spaces provide a compelling starting point since they imbue engineering applications: a serial-link robotic manipulator constructed of lower pairs between links comes to mind. Mathematical abstractions of physical phenomena, such as rigid motions ($\mathrm{SE}(n) = \mathrm{SO}(n) \times \mathbb{R}^n$), also highlight this point. Other application specific examples include the "occlusion-free configuration space" in visual servoing (see Sec. 4) [7, 8]. In each case, the systems are *topologically decoupled*, but *mechanically coupled*. Thus, a systematic set of tools for building NFs on Cartesian product spaces adds a key engineering tool to the toolbox.

1.1 Contribution

This paper presents a mathematically simple method by which to compose two NFs on two respective manifolds with boundary, to generate a new NF on the Cartesian product space. The composition law provides a control system designer with a set of gains to shape the resultant potential function, without imperiling the formal convergence and obstacle avoidance guarantees afforded by the NF methodology. Thus if the topology of the free configuration space can be expressed as the Cartesian product of several lower dimensional manifolds for which known NFs exists, a new NF for the free configuration space may be easily constructed from the constituent components.[1]

1.2 Related literature

Potential Fields and Energy Methods. Khatib introduced artificial potential functions for robot control in his Ph.D. dissertation over 20 years ago [11], and later employed potentials for dynamical obstacle avoidance [12]. Since 1980, several researchers have tried to repair inadequacies present in the early application of potential functions to robot control, most notably, the problem of local minima. In addition to addressing local minima, Koditschek and Rimon's NFs "lift" to second order dynamical settings while still ensuring essentially global convergence and obstacle avoidance [14, 16, 19, 20]. Lopes and Koditschek recently extended the use of NFs to the control of nonholonomic systems, in the context of perceptual constraints [17].

Other energy methods also round out the toolbox. For example, gyroscopic forces between multiple agents, each negotiating the same potential field, avoids mutual collisions [5], providing an elegant compliment to NFs. Also, the method of controlled Lagrangians [1,2] shapes the input to a system so that the closed loop behavior is that of a system with a different, desired Lagrangian; this is, in a sense, a generalization of potential shaping.

Composition. This paper presents a technique for controller composition: building up a large, integrated control systems based on relatively simple components. Burridge *et al.* describe a means by which to compose controllers sequentially [4]; roughly speaking, given a set of controllers, with known attractors and known domains of attraction, the resulting composition yields

[1] The constituent manifolds can be arbitrarily complicated, so long as an NF has been designed for each one.

a controller whose domain is the union of the constituent controllers. Their method applies to any dynamical system for which a set of constituent controllers (possessing certain technical requirements) exists. Sequential composition sits at a level above NFs, so the two may be integrated synergistically. For example, Conner et al. applied sequential composition to a collection of NF-like potentials for global robot navigation [6].

Parallel composition aims to allocate a small number of actuated degrees of freedom to manage the energy of multiple, independent dynamical systems. Early attempts at parallel composition, applied to robot juggling, succeeded empirically [3], leading to recent formal methods for parallel and interleaving composition by Klavins and Koditschek [13]. Their method abstracts each of n independent dynamical systems to its phase, which evolves on a copy of S^1. The total system evolves on the n-fold Cartesian product of S^1, T^n, where they construct a flow derived in part from a potential field. The systems they consider are topologically *and* (at least piecewise) mechanically "decoupled" (save the mechanical coupling of a shared actuation resource, such as the paddle in a juggling system [21]). An interesting extension of the current work would be to consider the parallel control of mechanically coupled systems evolving on "topologically decoupled" (i.e. cross product) spaces.

2 Control via Navigation Functions

This section summarizes the work on NFs developed by Rimon and Koditschek [14, 16, 19, 20].

2.1 Plant model

Consider a holonomically constrained, fully actuated mechanical system with known kinematics and suppose the configuration space is modeled by a compact, connected n-dimensional manifold with boundary, $\mathcal{Q}$. Let $(q, \dot{q}) \in T\mathcal{Q}$ denote the generalized positions and velocities on $\mathcal{Q}$. The equations of motion may be found using Lagrange's equations (see, for example, [9, 18]), namely

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}}L(q,\dot{q}) - \frac{\partial}{\partial q}L(q,\dot{q}) = u \tag{1}$$

where $L(q, \dot{q})$ is the Lagrangian, and u is a set of generalized force inputs. Assume that any external potentials (such as gravity) or non-viscous forces can be canceled by an appropriate feed-forward control term.

2.2 Task specification

Assume that any obstacles in the workspace are accounted for by the construction of $\mathcal{Q}$, so that for obstacle avoidance, trajectories must avoid crossing

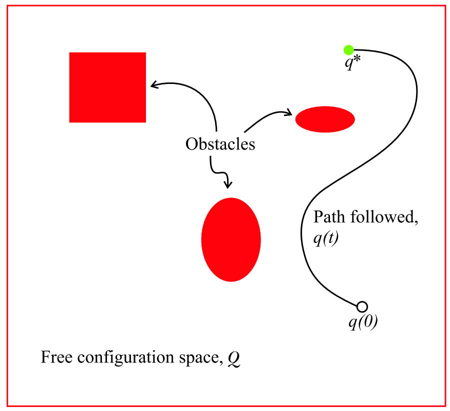

Fig. 1. An example of a free configuration space, $\mathcal{Q}$. In this case, $\mathcal{Q} \subset \mathbb{R}^2$ is a "square with three holes".

the boundary $\partial\mathcal{Q}$, for all $t \geq 0$ (see Fig. 1). The positioning objective is described in terms of a *goal*, q^*, in the interior of the domain, $\mathring{\mathcal{Q}}$. The task is to drive q to q^* asymptotically subject to (1) by an appropriate choice of u while remaining in $\mathcal{Q}$. Moreover, the basin of attraction $\mathcal{E}$ must include a dense subset of the zero velocity section of $T\mathcal{Q}$; this guarantees convergence from almost every initial zero velocity state $(q(0), \dot{q}(0)) = (q_0, 0)$ whose position component lies in $q_0 \in \mathcal{Q}$.

2.3 First order gradient systems

Let $\mathcal{Q}$ be an n-dimensional manifold with boundary, and consider the kinematic control system given by $\dot{q} = u$ where $u \in T\mathcal{Q}$ is a generalized velocity input. One possible control strategy involves following the gradient of a potential function $\varphi : \mathcal{Q} \to \mathbb{R}$, namely

$$\dot{q} = -\nabla\varphi(q). \tag{2}$$

A smooth scalar valued function whose Hessian matrix is non-singular at every critical point is called a *Morse* function [10]. Potential field controllers (2) arising from Morse functions impose global steady state properties that are particularly easy to characterize, as summarized in the following result.

Theorem 1 (Koditschek, [14]). *Let φ be a twice continuously differentiable Morse function on a compact Riemannian manifold, $\mathcal{Q}$. Suppose that $\nabla\varphi$ is transverse and directed away from the interior of $\mathcal{Q}$ on any boundary of that set. Then the negative gradient flow has the following properties:*

1. *$\mathcal{Q}$ is a positive invariant set;*
2. *the positive limit set of $\mathcal{Q}$ consists of the critical points of φ*
3. *there is a dense open set $\widetilde{\mathcal{Q}} \subset \mathcal{Q}$ whose limit set consists of the local minima of φ.*

2.4 Second order, damped gradient systems

The first order dynamical convergence results above do not directly apply to Lagrangian systems. This section reviews machinery to "lift" the gradient vector field controller (2) to one appropriate for the second order plant (1). Adding a linear damping term yields a nonlinear "PD" style feedback,[2]

$$u = -\nabla\varphi(q) - K_d\,\dot{q}, \tag{3}$$

and it follows that the total energy,

$$\eta(q,\dot{q}) = \varphi(q) + \kappa(q,\dot{q}),$$

where κ is the kinetic energy functional, is non-increasing [14].

Note that if the total initial energy exceeds the potential energy at some point on the boundary $\partial\mathcal{Q}$, a trajectory beginning within $\mathcal{Q}$ may intersect $\partial\mathcal{Q}$. Fortunately, further refining the class of potential functions will enable the construction of controllers for which the basin of attraction contains a dense subset of the zero velocity section of $T\mathcal{Q}$.

Definition 1. Let $\mathcal{Q}$ be an n-dimensional compact, simply connected manifold with boundary, and let $q^* \in \mathcal{Q}$ be distinct point. Let $\mathcal{C} \subset \partial\mathcal{Q}$, called the "corners" of $\mathcal{Q}$, be a nowhere dense subset. A functional $\varphi : \mathcal{Q} \to [0,1]$ is a *navigation function*, if it

1. is continuous on $\mathcal{Q}$ and C^2 on $\mathcal{Q} - \mathcal{C}$;
2. achieves its unique minimum of 0 at $q^* \in \mathring{\mathcal{Q}}$;
3. attains its maximal value of 1 uniformly on $\partial\mathcal{Q}$, the boundary of $\mathcal{Q}$ (assuming $\partial\mathcal{Q} \neq \emptyset$);
4. is Morse on $\mathcal{Q} - \mathcal{C}$.

This definition was adapted from [14,15,19]. In particular, [19] addresses the issue of manifolds with smooth interiors but possibly "sharp corners." For example, the square, $\mathcal{Q} = [-1,1] \times [-1,1]$, which has four sharp corners, admits an NF as will be seen in Sec. 4.

This notion of an NF, together with the observation that total energy decreases in a damped mechanical system, now yields the desired convergence result for the Lagrangian system (1).

Theorem 2 (Koditschek, [14]). *Given the system described by (1) subject to the control (3), almost every initial condition within the set*

$$\mathcal{E} = \{(q,\dot{q}) \in T\mathcal{Q} : \eta(q,\dot{q}) \leq 1\} \tag{4}$$

converges to $(q^*, 0)$ *asymptotically. Furthermore, transients remain within* $\mathcal{Q}$*, namely* $q(t) \in \mathcal{Q}$ *for all* $t \geq 0$*.*

Theorem 2 generalizes the kinematic global convergence of Theorem 1. Starting within $\mathcal{E}$ imposes a "speed limit" as well as a positional limit, since the total energy must be initially bounded [15].

[2] The allusion to PD control derives from the fact that near a minimum of φ, (3) reduces to $u \approx -K_p(q - q^*) - K_d\dot{q}$, where $K_p = \frac{\partial^2\varphi}{\partial q^2}(q^*)$ is the Hessian of φ.

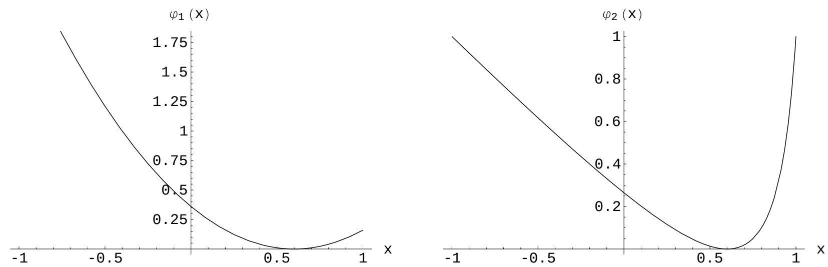

Fig. 2. An NF should evaluate uniformly at the boundary. In this case, $\mathcal{X} = [-1, 1]$, and φ_1 does not obtain the same value at $x = \pm 1$, whereas φ_2 does.

2.5 Invariance under diffeomorphism

A key ingredient in the mix of geometry and dynamics involves the realization that NFs are invariant to diffeomorphism [14]. This affords the introduction of geometrically simple model spaces and correspondingly simple NFs.

2.6 Why uniformly maximal at the boundary?

The requirement that an NF uniformly evaluate to a constant on the boundary is often overlooked. To illustrate this point, let $\mathcal{X} = [-1, 1]$, be the configuration manifold, and let $x^* = 0.6$ be the goal. The boundary, $\partial\mathcal{X}$, consists of two points, -1 and $+1$. As a candidate NF, one may naively consider

$$\varphi_1(x) = (x - 0.6)^2 \tag{5}$$

and note that $\nabla\varphi_1(x) = 2(x - 0.6)$, which has no local minima on $[-1, 1]$ except at the goal of $x^* = 0.6$. If the system begins at the left edge of the configuration manifold, it will have a total initial potential of $1.6^2 = 2.56$, but the potential barrier on the right edge is only 0.16. If the system is not highly over-damped, there are no guarantees that the right boundary will be avoided for all zero-velocity initial conditions within the domain $\mathcal{X}$. Thus, *second order safety* requires *boundary uniformity*. The NF

$$\varphi_2(x) = \frac{(x - 0.6)^2}{(1 - x^2) + (x - 0.6)^2}, \tag{6}$$

satisfies this requirement, as illustrated in Fig. 2.

3 Composing NFs on Cross Product Topologies

3.1 Construction

Consider two compact manifolds with boundary, $\mathcal{X}_1$ and $\mathcal{X}_2$, together with associated NFs, φ_1 and φ_2. Denote the Cartesian product space, $\mathcal{Q} = \mathcal{X}_1 \times \mathcal{X}_2$.

As a naive attempt construct a navigation function on $\mathcal{Q}$, consider the sum of NFs, $\varphi : \mathcal{Q} \to [0,1]$, given by

$$\varphi(q) = \tfrac{1}{2}(\varphi_1(x_1) + \varphi_2(x_2)),$$

where $q = (x_1, x_2)$. Such a function is Morse with unique minimum at $q^* = (x_1^*, x_2^*) \in \mathcal{Q}$, where $x_1^* \in \mathcal{X}_1$, $x_2^* \in \mathcal{X}_2$ are the respective goals of each NF. However, φ is *not* an NF on $\mathcal{Q}$. The problem arises because φ is not uniformly maximal on the boundary of $\mathcal{Q}$, given by the disjoint union of three components

$$\partial\mathcal{Q} = (\partial\mathcal{X}_1 \times \partial\mathcal{X}_2) \cup (\partial\mathcal{X}_1 \times \mathring{\mathcal{X}}_2) \cup (\mathring{\mathcal{X}}_1 \times \partial\mathcal{X}_2)\,.$$

For example, $\varphi(\cdot) < 1$ on $\partial\mathcal{X}_1 \times \mathring{\mathcal{X}}_2$, since $\varphi_2(\cdot) < 1$ on $\mathring{\mathcal{X}}_2$. Thus, the sum of two NFs, though possibly adequate for first-order kinematic systems, does not ensure safety with respect to the boundary when lifted to the second-order dynamic setting as described in Sec. 2.6.

Note that the function given by $q \mapsto (\varphi_1(x_1), \varphi_2(x_2))$, maps the boundary, $\partial\mathcal{Q}$, to the "top and right edges" of $[0,1]\times[0,1]$, namely $\{(1,\cdot)\}\cup\{(\cdot,1)\}$ and encodes the goal q^* at the point $(0,0)$. This motivates the following definition.

Definition 2. A *composition function* is a functional $\vartheta : [0,1]\times[0,1] \to [0,1]$ such that

1. $\vartheta(0,0) = 0$;
2. $\vartheta(1,\cdot) = \vartheta(\cdot,1) = 1$;
3. ϑ is C^2 everywhere but at the point $(1,1)$, where it is C^0.
4. ϑ is monotone increasing in both variables, i.e. $\frac{\partial\vartheta(z_1,z_2)}{\partial z_i} > 0$, $i = 1,2$.

We are now ready to compose two NFs.

Proposition 1 (Navigation Function Product). *Consider two compact manifolds with boundary, $\mathcal{X}_1$ and $\mathcal{X}_2$, together with associated NFs, φ_1 and φ_2, and a composition function, ϑ. The* navigation function product

$$\varphi = \varphi_1 \vee \varphi_2,$$

given by $\varphi(q) = \vartheta(\varphi_1(x_1), \varphi_2(x_2))$, is a navigation function on $\mathcal{Q} = \mathcal{X}_1 \times \mathcal{X}_2$, with unique global minimum at $q^* = (x_1^*, x_2^*) \in \mathcal{Q}$.

Proof. (Uniformly maximal) Since ϑ is a composition function, φ achieves the value of 1 exactly when either φ_1 and/or φ_2 achieve a value of 1, which comprises the entire boundary $\partial\mathcal{Q}$.

(Corners are nowhere dense) Let $\mathcal{C}_i \subset \partial\mathcal{X}_i$, $i = 1,2$ be the sets of corners associated with each respective manifold. Consider the set $\mathcal{C} \subset \partial\mathcal{Q}$, comprised of three disjoint components

$$\mathcal{C} = \mathcal{J}_1 \cup \mathcal{J}_2 \cup \mathcal{J}_3, \text{ where } \begin{cases} \mathcal{J}_1 = \mathcal{C}_1 \times \mathcal{X}_2 \\ \mathcal{J}_2 = \mathcal{X}_1 \times \mathcal{C}_2 \\ \mathcal{J}_3 = \partial\mathcal{X}_1 \times \partial\mathcal{X}_2\,. \end{cases}$$

This set comprises the set of corners of $\mathcal{Q}$, and as we will see, is the exact set on which $\varphi = \varphi_1 \vee \varphi_2$ is not smooth. Note that even if $\mathcal{X}_1$ and $\mathcal{X}_2$ are both corner-free, the Cartesian product $\mathcal{X}_1 \times \mathcal{X}_2$ has corners $\mathcal{J}_3$.

By hypothesis, $\mathcal{C}_i \subset \partial\mathcal{X}_i$, $i = 1, 2$ are nowhere dense, therefore $\mathcal{C}_1 \times \mathcal{X}_2 \subset \partial\mathcal{Q}$ (and vice versa) are nowhere dense. Moreover, $\partial\mathcal{X}_1 \times \partial\mathcal{X}_2$ is nowhere dense on $\partial\mathcal{Q}$. The finite union of nowhere dense sets is nowhere dense.

(Continuity) Note that φ_i is C^2 on $\mathcal{X}_i - \mathcal{C}_i$, $i = 1, 2$ and ϑ is C^2 on $[0,1]\times[0,1]-\{(1,1)\}$. By construction, $\mathcal{C}$ comprises exactly the "bad regions" where these functions are not smooth, and thus φ is C^2 on $\mathcal{Q} - \mathcal{C}$. Moreover, φ is the composition of continuous functions and is therefore continuous (C^0) everywhere on $\mathcal{Q}$.

(Morse) Note that

$$\nabla\varphi = \left.\begin{bmatrix} \frac{\partial\vartheta(z_1,z_2)}{\partial z_1}\nabla\varphi_1 \\ \frac{\partial\vartheta(z_1,z_2)}{\partial z_2}\nabla\varphi_2 \end{bmatrix}\right|_{z_1=\varphi_1, z_2=\varphi_2}$$

which is well defined everywhere except on $\mathcal{C}$. By Definition 2, $\frac{\partial\vartheta(z_1,z_2)}{\partial z_i} > 0$, $i = 1, 2$, so on $\mathcal{Q} - \mathcal{C}$ we have

$$\nabla\varphi(q) = 0 \iff \nabla\varphi_1(x_1) = 0 \text{ and } \nabla\varphi_2(x_2) = 0\,.$$

Thus, the critical points of φ are given simply by all combinations of critical points of φ_1 and φ_2. The Hessian at a critical point[3] is given by

$$\frac{\partial^2\varphi}{\partial q^2} = \left.\begin{bmatrix} \frac{\partial\vartheta(z_1,z_2)}{\partial z_1}\frac{\partial^2\varphi_1}{\partial x_1^2} & 0 \\ 0 & \frac{\partial\vartheta(z_1,z_2)}{\partial z_2}\frac{\partial^2\varphi_2}{\partial x_2^2} \end{bmatrix}\right|_{z_1=\varphi_1, z_2=\varphi_2}. \tag{7}$$

Thus, the Hessian matrix evaluated at a critical point (7) is block diagonal with the positively scaled Hessians of each constituent NF on the diagonal. Thus, since the constituent Hessians are nondegenerate at a critical point, then φ is also nondegenerate.

(Unique minimum) The critical point $q^* = (x_1^*, x_2^*)$ corresponding to a minimum of both constituent potential functions is also a minimum. Moreover, it is the global minimum since $\varphi(q) = 0$ iff $\varphi_i(x_i) = 0$, $i = 1, 2$, which is only true at q^*. By inspection of the Hessian, all other others critical points are saddles and maxima. $\square$

3.2 Designing composition functions

The composition functions presented below, while by no means exhaustive, have certain convenient properties, such as associativity, tunability, and successive compositions reduce to a single, computationally simple expression.

[3] This expression is not valid away from the critical points, since it explicitly uses the fact that $\nabla\varphi_i = 0$, $i = 1, 2$.

Consider the "squashing diffeomorphism" $\sigma : [0,\infty) \to [0,1)$, defined by

$$\sigma(\alpha) := \frac{\alpha}{1+\alpha} \tag{8}$$

and the function, $\chi : [0,1) \times [0,1) \to [0,\infty)$, given by

$$\chi(z_1, z_2) = \frac{z_1}{1-z_1} + \frac{z_2}{1-z_2}. \tag{9}$$

Finally, define $\vartheta : [0,1] \times [0,1] \to [0,1]$ by

$$\vartheta(z_1, z_2) := \begin{cases} \sigma \circ \chi(z_1, z_2), & \text{when } z_1 < 1 \text{ and } z_2 < 1 \\ 1, & \text{otherwise.} \end{cases} \tag{10}$$

To see that this function satisfies Definition 2 note that

$$\vartheta(z_1, z_2) = \frac{z_1 + z_2 - 2z_1 z_2}{1 - z_1 z_2} \tag{11}$$

everywhere on $[0,1] \times [0,1]$, except at $z_1 = z_2 = 1$ (i.e. the "upper right corner"). In particular, note that the above expression evaluates to 1 when either $z_1 = 1$ or $z_2 = 1$ (but the expression is not well defined for $z_1 = z_2 = 1$). Nevertheless, the limit

$$\lim_{z_1, z_2 \to 1} \frac{z_1 + z_2 - 2z_1 z_2}{1 - z_1 z_2} = 1$$

and therefore ϑ is continuous, and well defined on its domain $[0,1] \times [0,1]$. Furthermore, the Jacobian of ϑ,

$$\frac{\partial \vartheta(z_1, z_2)}{\partial z} = \frac{1}{(1 - z_1 z_2)^2} \begin{bmatrix} (1-z_2)^2 & (1-z_1)^2 \end{bmatrix},$$

is well defined except when $z_1 = 1$ or $z_2 = 1$, and can be easily extended by taking the limit as $z_1 \to 1$ or $z_2 \to 1$ (but not both) to yield

$$\lim_{z_1 \to 1} \left(\frac{\partial \vartheta(z_1, z_2)}{\partial z} \right) = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \lim_{z_2 \to 1} \left(\frac{\partial \vartheta(z_1, z_2)}{\partial z} \right) = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

As can be seen, the Jacobian is discontinuous at the point $z = (1,1)$. It is also straight forward to compute the Hessian matrix, which is well defined everywhere on the domain $[0,1] \times [0,1]$ except at $z = (1,1)$.

Remark 1 (Semigroup property). By Proposition 1, if φ_1 and φ_2 are, respectively, NFs on compact manifolds with boundary, $\mathcal{X}_1$ and $\mathcal{X}_2$, then $\varphi_1 \vee \varphi_2$ is an NF on $\mathcal{X}_1 \times \mathcal{X}_2$, and thus $\vee$ is closed. Moreover, Cartesian products of spaces are associative, i.e.

$$(\mathcal{X}_1 \times \mathcal{X}_2) \times \mathcal{X}_3 = \mathcal{X}_1 \times (\mathcal{X}_2 \times \mathcal{X}_3).$$

Finally, the navigation product is associative because ϑ is associative, namely $\vartheta(z_1, \vartheta(z_2, z_3)) = \vartheta(\vartheta(z_1, z_2), z_3)$. This can be verified by direct algebraic substitution. Thus, the set NFs on compact manifolds with boundary forms a semigroup. □

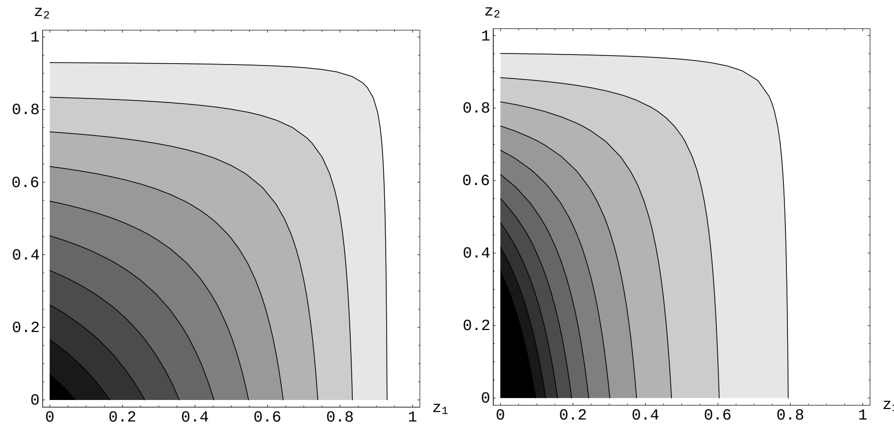

Fig. 3. Contour plots for different composition functions. *Left:* untuned ϑ in (10). *Right:* tuned ϑ_{k} in (12), with $k = (1, 5)$.

Remark 2 (Fold). Let $\mathcal{X}_i$, $i = 1, \ldots, n$ be n manifolds with boundary and let φ_i be their respective NFs with goals x_i, $i = 1, \ldots, n$. Using the composition function ϑ, let $\varphi = \varphi_1 \vee \varphi_2 \cdots \vee \varphi_n$, $\mathcal{Q} = \mathcal{X}_1 \times \mathcal{X}_2 \times \cdots \mathcal{X}_n$, and let $q = (x_1, x_2, \ldots, x_n) \in \mathcal{Q}$. Then[4]

$$\varphi(q) = \sigma\left(\sum_{i=1}^{n} \frac{\varphi_i(x_i)}{1 - \varphi_i(x_i)} \right).$$

(Proof by induction). By definition, this is true for $n = 2$. Suppose it is true for $n = k$, and let $\varphi^k := \varphi_1 \vee \varphi_2 \cdots \vee \varphi_k$ be the navigation product of the first k functions. Dropping the explicit dependence on the x_i's, from (11) we have that

$$\varphi^{k+1} = \vartheta\big(\varphi^k, \varphi_{k+1}\big) = \frac{\varphi_{k+1} + \sigma\Big(\sum_{i=1}^{k} \frac{\varphi_i}{1-\varphi_i} \Big) - 2\varphi_{k+1}\sigma\Big(\sum_{i=1}^{k} \frac{\varphi_i}{1-\varphi_i} \Big)}{1 - \varphi_{k+1}\sigma\Big(\sum_{i=1}^{k} \frac{\varphi_i}{1-\varphi_i} \Big)}$$

which, upon multiplying numerator and denominator by $1 + \sum_{i=1}^{k} \frac{\varphi_i}{1-\varphi_i}$, and simplifying yields the desired result. $\square$

Tunable composition function. While the NF framework affords certain formal guarantees, such as dynamical obstacle avoidance and essentially global convergence, one of the practical weaknesses of the existing NF literature concerns the tunability of NF-based controllers. In (3), the damping gain, K_d, comprises a set of a free design parameters, but there are no explicit hooks for

[4] One must use care on $\partial\mathcal{Q}$, e.g. when $\varphi_i = 1$, but at these points $\varphi(q) = 1$.

tuning the potential function, unless the designer builds such hooks directly into the NF. The composition function formed by replacing χ (9) with

$$\chi_k(z_1, z_2) = k_1 \frac{z_1}{1 - z_1} + k_2 \frac{z_2}{1 - z_2}$$

introduces a set of tunable gains $k = (k_1, k_2)$. As long as $k_1, k_2 > 0$, it is readily shown that the resulting composition function

$$\vartheta_{\mathrm{k}}(z_1, z_2) := \begin{cases} \sigma \circ \chi_k(z_1, z_2), & \text{when } z_1 < 1 \text{ and } z_2 < 1 \\ 1, & \text{otherwise.} \end{cases} \tag{12}$$

satisfies Definition 2.

With appropriate choice of gains at each stage of composition, the $\vee$ is associative as before. For example

$$\vartheta_{(k_1,1)}\big(z_1, \vartheta_{(k_2,k_3)}(z_2, z_3)\big) = \vartheta_{(1,k_3)}\big(\vartheta_{(k_1,k_2)}(z_1, z_2), z_3\big),$$

and $\varphi = \varphi_1 \vee \varphi_2 \cdots \vee \varphi_n$ is given by

$$\varphi(q) = \sigma\left(\sum_{i=1}^{n} k_i \frac{\varphi_i(x_i)}{1 - \varphi_i(x_i)}\right).$$

4 Examples

We consider a few simple examples of the navigation product.

4.1 Cross product of a circle and an interval

Note that $\varphi_1(\theta) = (1 - \cos\theta)/3$ is an NF on S^1 with a goal at $\theta = 0$. The maximum value of φ_1 is chosen to be $2/3$, rather 1, since there is no boundary on S^1. The function $\varphi_2(x) = x^2$ is an NF on $[-1, 1]$, with a goal at $x = 0$. Let $\mathcal{Q} = \mathrm{S}^1 \times [-1, 1]$, and let $q = (\theta, x) \in \mathcal{Q}$. The navigation product $\varphi = \varphi_1 \vee \varphi_2 : \mathcal{Q} \to [0, 1]$ is given by

$$\varphi(q) = (\varphi_1 \vee \varphi_2)(\theta, x) = \frac{1 + x^2 + \left(-1 + 2\,x^2\right)\cos\theta}{3 - x^2 + x^2\cos\theta}.$$

A 3D plot is shown in Figure 4.

4.2 Cross product of intervals

Let $\mathcal{X}_i = [\alpha_i, \beta_i]$, for some $\alpha_i, \beta_i \in \mathbb{R}$, be two intervals and consider a different NF, $\varphi_i : \mathcal{X}_i \to [0, 1]$, for each

$$\varphi_i(x) = \frac{(x - q_i^*)^2}{(\beta_i - x)(x - \alpha_i) + (x - q_i^*)^2}, \quad i = 1, 2, \tag{13}$$

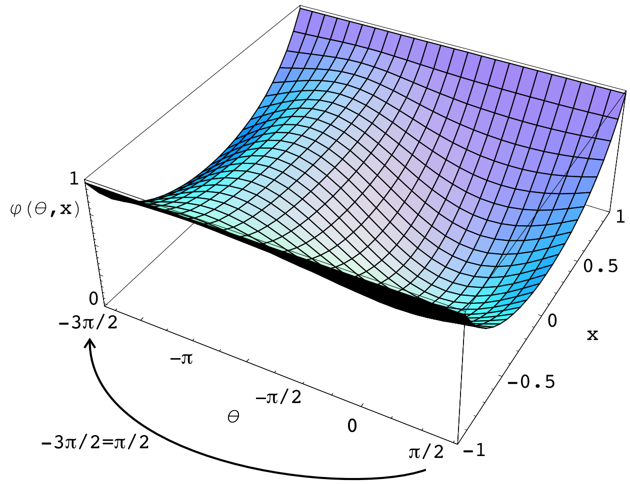

Fig. 4. A 3D plot of an NF on $S^1 \times [-1,1]$ derived from two simple NFs on S^1 and $[-1,1]$, respectively. The endpoints of the θ axis are identified. There are two critical points, a saddle and a minimum.

with goals at q_i^*, i=1,2. Let $\mathcal{Q} = \mathcal{X}_1 \times \mathcal{X}_2$, and let $q = (q_1, q_2) \in \mathcal{Q}$. Consider the naive NF candidate given by

$$\varphi_{\text{naive}}(y) = \tfrac{1}{2}(\varphi_1(q_1) + \varphi_1(q_2)).$$

As can be seen from Fig. 6, this naive construction is not an NF, whereas the navigation product $\varphi = \varphi_1 \vee \varphi_1$ is an NF. In the example plot, the configuration space is $\mathcal{Q} = [-\frac{\pi}{4}, \frac{\pi}{4}]^2$, with the goal location $q^* = (-\frac{3\pi}{20}, 0)$.

Note that the corners of $\mathcal{Q}$ are the four points $(-\frac{\pi}{4}, -\frac{\pi}{4})$, $(-\frac{\pi}{4}, \frac{\pi}{4})$, $(\frac{\pi}{4}, \frac{\pi}{4})$, $(\frac{\pi}{4}, -\frac{\pi}{4})$. Similarly, for the Cartesian product of three intervals, e.g. $\mathcal{Q} = [-1, 1]^3$, the corners are the edges of the cube.

4.3 Double pendulum

We simulated a two link, revolute-revolute mechanical system, as shown in Fig. 5, to illustrate the consequences of naive artificial potential function design. We assumed joint limits of $\pm\frac{\pi}{4}$, and thus $\mathcal{Q} = [-\frac{\pi}{4}, \frac{\pi}{4}]^2$. We chose simulated link lengths of $\ell_1 = \ell_2 = 1$ and masses of $m_1 = m_2 = 1$ located at the end of each respective link. Finally, we chose a diagonal damping matrix with diagonal elements given by $(1.5, 0.5)$.

For each DOF, we computed the NF, φ_i, $i = 1, 2$, as given in (13), where the goals were chosen to be $(q_1^*, q_2^*) = (-\frac{3\pi}{20}, 0)$. We applied the control law given by (3) for both the "naive" potential function, $\varphi_{\text{naive}} = \frac{1}{2}(\varphi_1 + \varphi_2)$, and a true NF given by $\varphi = \varphi_1 \vee \varphi_2$.

If the system were kinematic (2), or the links were mechanically uncoupled, then control based on, φ_{naive}, would guarantee that all zero-velocity

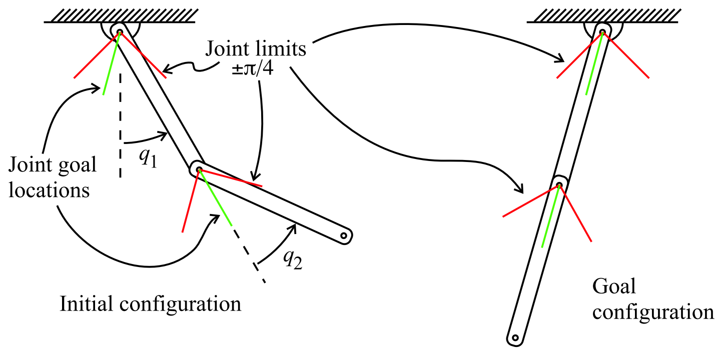

Fig. 5. A "double pendulum."

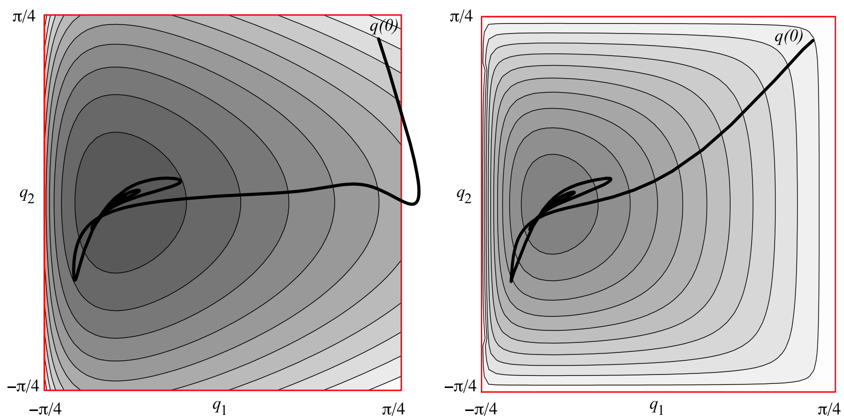

Fig. 6. Contour plots of two candidate NFs on the space $\mathcal{Q} = [-\frac{\pi}{4}, \frac{\pi}{4}]^2$. The configuration space trajectories resulting from a zero velocity initial condition,$(q(0), 0) \in T\mathcal{Q}$, of the double pendulum system shown in Fig. 5, subject to the "gradient + damping" feedback in (3) are shown by the bold curves. *Left.* The potential function, $\varphi_{\text{naive}}(q_1, q_2) = \frac{1}{2}(\varphi_1(q_1) + \varphi_2(q_2))$, is not an NF, because it is not uniformly maximal on the boundary, and thus the trajectory crosses the boundary. *Right.* The navigation product, $\varphi = \varphi_1 \vee \varphi_2$, from Proposition 1 is an NF, thus ensuring safety with respect to the boundary.

initial conditions within the free configuration space, $\mathcal{Q}$, safely converge. However, the mechanical coupling between the links renders the behavior undesirable since the first link violates its joint limit, as shown in Fig. 6.

4.4 Occlusion-Free visual servoing

We turn to a slightly more complex example. Cowan and Chang [7] showed that the set of all configurations of a perspective projection camera with a limited field of view, that keeps a specific visual target completely in view, is diffeomorphic to the model space

$$\mathcal{Q} = \mathcal{P} \times [-1,1] \times \mathcal{U} \tag{14}$$

where $\mathcal{P} := \{R \in \mathrm{SO}(3) : r_1 \cdot e_1 \geq \cos\psi\}$, $\mathcal{U} = \overline{\mathbf{D}^2} = \{u \in \mathbb{R}^2 : u_1^2 + u_2^2 \leq 1\}$ is a closed, planar unit disk, and $\psi \in (0, \pi/2)$ is a constant.

The basic construction of NFs on each configuration component follows Rimon and Koditschek [16]. Define two basic building blocks: a *goal function*, $\gamma : \mathcal{X} \to [0,\infty)$, possessing a unique minimum at a *goal point*, $x^* \in \mathcal{X}$, and an *obstacle function*, $\beta : \mathcal{X} \to [0,\infty)$ that vanishes (only) on the boundary $\partial\mathcal{X}$, which is treated as an obstacle set. These two building blocks are assembled to create a function $\varphi : \mathcal{X} \to [0,1]$:

$$\varphi := \frac{\gamma}{\gamma + \beta}. \tag{15}$$

This construction ensures that φ is uniformly maximal on $\partial\mathcal{X}$. The critical point structure must be verified on a case-by-case basis.

1. *NF on* $\mathcal{P}$. Let $R = \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix} \in \mathrm{SO}(3)$. Define goal and obstacle functions, respectively, as

$$\gamma_1(R) := \mathrm{trace}\left(\left(I - RR^{*T}\right)\Lambda\right), \ \beta_1(R) := r_1^T e_1 - \cos\psi \tag{16}$$

where Λ is a diagonal matrix with three distinct positive eigen values, $\lambda_1, \lambda_2, \lambda_3$. Let $\varphi_1 := \gamma_1/(\gamma_1 + \beta_1)$ as described above. As shown in [7], when $R^* = I$, the function $\varphi_1 : \mathcal{P} \to \mathbb{R}$, has a unique global minimum at $R = I$, and any other critical points are non-degenerate saddles and maxima. Therefore, the function φ_1, for a goal location of $R^* = I$, is an NF.
2. *NF on* $[-1,1]$. Define goal and obstacle functions, respectively, as

$$\gamma_2(\zeta) := (\zeta - \zeta^*)^2, \ \beta_2(\zeta) := 1 - \zeta^2, \tag{17}$$

where $\zeta^* \in (-1,1)$ is the goal point. One can show that φ_2, defined in (15) is an NF. (This is equivalent to (13) with $\alpha = -1$ and $\beta = 1$.)
3. *NF on* $\mathcal{U}$. The manifold $\mathcal{U} := \{u \in \mathbb{R}^2 : \|u\| \leq 1\}$ is the simplest form of a "sphere world," as defined by Koditschek and Rimon [16]. Define goal and obstacle functions, respectively, as

$$\gamma_3(u) := \|u - u^*\|^2, \ \beta_3(u) := 1 - u_1^2 - u_2^2, \tag{18}$$

where $u^* \in \mathbf{D}^2$ (the interior of $\mathcal{U}$). The function φ_3, as defined in (15), is an NF [16].

Let $q = (R, \zeta, u)$. Given the NFs φ_1, φ_2 and φ_3 defined on their respective manifolds $\mathcal{P}$, $[-1,1]$ and $\mathcal{U}$, then the function $\varphi = \varphi_1 \vee \varphi_2 \vee \varphi_3$, given by

$$\varphi(q) := \sigma\left(k_1 \frac{\varphi_1(R)}{1 - \varphi_1(R)} + k_2 \frac{\varphi_2(\zeta)}{1 - \varphi_1(\zeta)} + k_3 \frac{\varphi_3(u)}{1 - \varphi_3(u)}\right), \tag{19}$$

is an NF on the cross product space, $\mathcal{Q} = \mathcal{P} \times [-1,1] \times \mathcal{U}$.

5 Conclusions

The goal of this paper is to render the machinery of NFs more useful. Typically, each new space – even if just a simple Cartesian product of other spaces with known NFs – has required the construction of a new NF. This paper enables a designer to construct an obstacle avoiding "spring law" for each separate configuration component, and then the navigation product, $\vee$, stitches the spring laws together so that dynamical obstacle avoidance is maintained even for coupled mechanical systems. The technique makes no extra assumptions on the topology or geometry of the underlying spaces than those already required for the NF literature. Moreover, the composition law has the virtue of being associative, tunable, and computational simple even for large numbers of successive cross products.

For real systems, a significant amount of "hand" design may still be necessary, but the design process is vastly simplified by decomposing the problem into (topologically) separate pieces, and then combining the results together with the navigation function product.

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