# Navigation Functions on Cross Product Spaces

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***Abstract*— Given two compact, connected manifolds with corners, and a navigation function (NF, a refined artificial potential function) on each manifold, this paper presents a simple composition law that yields a new NF on the cross product space. The method provides tunable "hooks" for shaping the new potential function while still guaranteeing obstacle avoidance and essentially global convergence. The composition law is associative, and successive compositions fold into a single, computational simple expression, enabling the practical construction of NFs on the Cartesian product of several manifolds.**

## I. Introduction

The gradient vector field of a properly designed artificial potential function can steer a robot to a goal, while avoiding obstacles along the way. Adding a damping term to flush out any unwanted kinetic energy generalizes this approach to the second-order setting, since total energy always decreases in damped mechanical systems [1], [2].

The problem of spurious minima and safety for second-order plants lead Koditschek [2] to introduce navigation functions (NFs), a refined notion of artificial potential functions. He showed that every smooth compact connected manifold with boundary, $\mathcal{M}$, admits a smooth NF, $\varphi : \mathcal{M} \to \mathbb{R}$ [2]. Thus, given a fully actuated Lagrangian system evolving on a such a configuration manifold, the machinery of NFs "solves" the global dynamical control and obstacle avoidance problem [3]. However, constructing NFs for arbitrary manifolds remains an art: each new model space requires a handcrafted NF. Rimon and Koditschek developed several constructions for special model spaces including "sphere" worlds [4] and "star" worlds [5]. Recent applications of NFs include formation control [6], [7] and control of nonholonomic systems [8]. Also, two recent composition schemes involve covering the configuration space with NF-like potentials for global robot navigation [9], [10].

While a general methodology for constructing NFs seems unlikely, an "NF designers toolbox" for families of configuration spaces of practical interest seems plausible. Cartesian product spaces provide a compelling starting point since they imbue engineering applications: a serial-link robotic manipulator constructed of lower pairs between links comes to mind. Mathematical abstractions of physical phenomena, such as rigid motions ($\mathrm{SO}(n) \times \mathbb{R}^n$), also highlight this point. Other application-specific examples include the "occlusion-free configuration space" in visual servoing [11]. In each case, the systems are topologically decoupled, but mechanically coupled. Thus, a set of tools for building NFs on Cartesian product spaces adds a useful engineering tool to the toolbox.

This note presents a simple method by which to compose two NFs on two respective manifolds with corners, to generate a new NF on the Cartesian product space. The composition law provides a control system designer with a set of gains to shape the resultant potential function, without imperiling the formal convergence and obstacle avoidance guarantees afforded by the NF methodology. Thus if the topology of the free configuration space can be expressed as the Cartesian product of several lower dimensional manifolds for which known NFs exists, a new NF for the free configuration space may be easily constructed from the constituent components; the constituent pieces are arbitrary manifolds with corners – so long as an NF has already been designed for each one!

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## II. Control via Navigation Functions

This section briefly summarizes the work on NFs developed by Rimon and Koditschek [2]–[5]. Specifically, I restate key results of Koditschek [2], modestly extending them where necessary to the present context of manifolds with corners.

### A. *Configuration Space*

Our interest is in Cartesian product spaces, e.g. the solid cube $[-1, 1]^3$. Such manifolds often have "sharp corners" (e.g. the vertices and edges of the cube), even when constructed of smooth constituent manifolds with boundary. Fortunately, the basic construction of NFs proposed by Koditschek and Rimon almost two decades ago for manifolds with boundary works in the context of manifolds with corners with very little modification, as shown below.

Following pieces of Handron [12] and Vakhrameev [13], I define a manifold with corners as follows. A chart on a space $\mathcal{Q}$ is a map $\varphi : U \to \mathbb{R}^n$ such that $\varphi$ maps open sets $U \subset \mathcal{Q}$ homeomorphically onto a (solid) convex polyhedral cone. An $n$-dimensional manifold with corners is a topological space $\mathcal{Q}$, together with an a (maximal) atlas $\Phi$ of charts $\varphi_i : U_i \to \mathbb{R}^n$, $i \in \Gamma$, that such that the union $\{U_i\}_{i\in\Gamma}$ is an open cover of $\mathcal{Q}$. $(\mathcal{Q}, \Phi)$ is a $C^r$ manifold with corners if the mappings

$$\varphi_j \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j) \tag{1}$$

are $C^r$ (that is, they can be extended to $C^r$ maps between open sets of $\mathbb{R}^n$). The boundary $\partial\mathcal{Q}$ is comprised of points that are boundary points for some chart.

Note that a manifold with boundary [14] is a special case of a manifold with corners.

A Riemannian metric on a $C^r$ manifold with corners [12] $\mathcal{Q}$ is a symmetric bilinear form $< \cdot, \cdot >_q$ on $T_q\mathcal{Q}$ such that $< \frac{\partial}{\partial q_i}, \frac{\partial}{\partial q_j} > : \varphi(U) \to \mathbb{R}$ is $C^r$ (again, it can be extended to a $C^r$ function on an open set in $\mathbb{R}^n$) for any local coordinate chart $(\varphi, U)$.

Consider two $C^r$ manifolds with corners $(\mathcal{X}, \Phi)$, $(\mathcal{Y}, \Psi)$ of dimension $n$ and $m$ and with charts $\varphi_i : U_i \to \mathbb{R}^n$, $i \in \Gamma$ and $\psi_j : V_j \to \mathbb{R}^m$, $j \in \Lambda$ respectively. The Cartesian product $(\mathcal{X} \times \mathcal{Y}, \Theta)$ is a manifold with corners of dimension $n + m$, where the atlas (also called a differential structure) $\Theta$ contains all charts of the form $\vartheta_{i,j} : U_i \times V_j \to \mathbb{R}^{n+m}$ where $\vartheta_{i,j}(x, y) := (\varphi_i(x), \psi_j(y))$ map open sets of $U_i \times V_i$ homeomorphically onto the Cartesian product of two solid polyhedral cones in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, which is itself a solid polyhedral cone in $\mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m}$.

Clearly, for two charts $\vartheta_i, \vartheta_j \in \Theta$, the mapping $\vartheta_j \vartheta_i^{-1} = \varphi_j \varphi_i^{-1} \times \psi_j \psi_i^{-1}$ is $C^r$, since the differentiable structures on $\mathcal{X}$ and $\mathcal{Y}$ are $C^r$. Thus, by construction, the resulting object $(\mathcal{X} \times \mathcal{Y}, \Theta)$ is a $C^r$ manifold with corners. Hirsch [14] defines the Cartesian product of two compact manifolds without boundary in a similar manner.

### B. Plant model

Consider a holonomically constrained fully actuated mechanical system with known kinematics and suppose the configuration space is modeled by a compact, connected $n$-dimensional manifold with corners, $\mathcal{Q}$. Let $(q, \dot{q}) \in T\mathcal{Q}$ denote the generalized positions and velocities on $\mathcal{Q}$ (the tangent bundle is defined similarly as for manifolds with boundary [12]). The equations of motion may be found using Lagrange's equations, namely

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}} L(q, \dot{q}) - \frac{\partial}{\partial q} L(q, \dot{q}) = u \tag{2}$$

where $L(q, \dot{q})$ is the Lagrangian, and $u$ is a set of generalized force inputs. Assume that any external potentials (such as gravity) or non-viscous forces can be canceled by an appropriate feed-forward control term.

### C. Task specification

Assume that any obstacles in the workspace are accounted for by the construction of $\mathcal{Q}$, so that for obstacle avoidance, trajectories must avoid crossing the boundary $\partial\mathcal{Q}$, for all $t \geq 0$. The positioning objective is described in terms of a *goal*, $q^*$, in the interior of the domain, $\mathring{\mathcal{Q}}$. The task is to drive $q$ to $q^*$ asymptotically subject to (2) by an appropriate choice of $u$ while remaining in $\mathcal{Q}$. Moreover, the basin of attraction $\mathcal{E}$ must include a dense subset of the zero velocity section of $T\mathcal{Q}$; this guarantees convergence from almost every initial zero velocity state $(q(0), \dot{q}(0)) = (q_0, 0)$ whose position component lies in $q_0 \in \mathcal{Q}$.

### D. Navigation Functions on Manifolds with Corners

A functional $\varphi \in C^r[\mathcal{Q}, [a, b]]$ is *Morse* if it all its critical points are nondegenerate (that is, the Hessian is nonsingular at each critical point) [14], and *admissible* if $\partial\mathcal{Q} = \varphi^{-1}(a) \cup \varphi^{-1}(b)$. The following definition is equivalent to Koditschek's definition of NFs [2], except that he required differentiability on the boundary.

*Definition 1 (Adapted from [2]):* Let $\mathcal{Q}$ be an $n$-dimensional compact, connected manifold with corners, and let $q^* \in \mathcal{Q}$ be distinct point. A functional $\varphi : \mathcal{Q} \to [0, 1]$ is a *navigation function* (NF), if it

1) is Morse on $\mathring{\mathcal{Q}}$;
2) is $C^r$ on $\mathring{\mathcal{Q}}$, $r \geq 2$;
3) is admissible over the interval $[-\epsilon, 1]$, $\epsilon \in \mathbb{R}^+$, with $\partial\mathcal{Q} = \varphi^{-1}(1)$ (or $\partial\mathcal{Q} = \emptyset$);
4) achieves its unique minimum of 0 at $q^* \in \mathring{\mathcal{Q}}$.

### E. Second order, damped gradient systems

One possible control strategy involves following the gradient of an NF, $\varphi : \mathcal{Q} \to [0, 1]$, together with a damping term, yielding a nonlinear "PD" style feedback,[1]

$$u = -\nabla\varphi(q) - K_d\, \dot{q}. \tag{3}$$

It follows that the total energy,

$$\eta(q, \dot{q}) = \varphi(q) + \kappa(q, \dot{q}),$$

where $\kappa$ is a kinetic energy functional, is non-increasing [2]. Assume henceforth that $\varphi$ is $C^r$ on $\mathring{\mathcal{Q}}$, $\kappa$ is $C^r$ on $T\mathring{\mathcal{Q}}$, and hence $\eta$ is $C^r$ on $T\mathring{\mathcal{Q}}$.

Note that if the total initial energy exceeds the potential energy at some point on the boundary $\partial\mathcal{Q}$, a trajectory beginning within $\mathcal{Q}$ may intersect $\partial\mathcal{Q}$. Fortunately, the definition of NFs enables the construction of controllers for which the basin of attraction contains a dense subset of the zero velocity section of $T\mathcal{Q}$, as described by Koditschek for manifolds with *smooth* boundary.

*Theorem 1 (Koditschek [2]):* Let $\mathcal{Q}$ be a compact Riemannian manifold with boundary. Let $\varphi : \mathcal{Q} \to [0, 1]$ be an NF that is, additionally, $C^r$ on the boundary $\partial\mathcal{Q}$. Given the system described by (2) subject to the control (3), almost every initial condition within the set[2]

$$\mathcal{E} = \{(q, \dot{q}) \in T\mathcal{Q} : \eta(q, \dot{q}) < 1\} \tag{4}$$

converges to $(q^*, 0)$ asymptotically. Furthermore, transients remain within $\mathcal{Q}$, namely $q(t) \in \mathcal{Q}$ for all $t \geq 0$.

We may now extend Koditschek's result to manifolds with corners.

*Corollary 1:* Let $\mathcal{Q}$ be a compact Riemannian manifold with corners, and let $\varphi : \mathcal{Q} \to [0, 1]$ be an NF. Then Theorem 1 holds.

*Proof:* Each set $\varphi^{-1}(1 - \epsilon)$, such that $(1 - \epsilon) \in (0, 1)$ is not a critical value of $\varphi$, is a smooth regular level surface. Therefore, $\mathcal{Q}_\epsilon := \{q \in \mathcal{Q} : \varphi(q) \leq 1 - \epsilon\}$ is a compact submanifold $\mathcal{Q}_\epsilon \subset \mathring{\mathcal{Q}}$ with smooth boundary $\partial\mathcal{Q}_\epsilon = \varphi^{-1}(1 - \epsilon)$. Up to a scale factor $\varphi|_{\mathcal{Q}_\epsilon}$ is an NF that is, in addition, $C^r$ on the boundary $\partial\mathcal{Q}_\epsilon$ (since the boundary is wholly contained in $\mathring{\mathcal{Q}}$, where $\varphi$ is smooth by definition).

Now, let $\mathcal{E}_\epsilon = \{(q, \dot{q}) \in T\mathcal{Q}_\epsilon : \eta(q, \dot{q}) < 1 - \epsilon\} \subset T\mathring{\mathcal{Q}}$. Then Theorem 1 implies that almost every initial condition within $\mathcal{E}_\epsilon$ converges to $(q^*, 0)$ asymptotically, and no motions $q(t)$ cross $\partial\mathcal{Q}_\epsilon$, for all sufficiently small $\epsilon$ (that is, all those epsilon for which $1 - \epsilon$ is greater than the largest critical value of $\varphi$). Since $\eta$ is $C^r$ on $T\mathring{\mathcal{Q}}$, then for every point $(q, \dot{q}) \in \mathcal{E}$, there exists $\epsilon > 0$ such that $(q, \dot{q}) \in \mathcal{E}_\epsilon$, implying that almost all initial conditions within $\mathcal{E}$ converge to $(q^*, 0)$, while ensuring that motions $q(t)$ do not cross the boundary $\partial Q$. ■

[1] The allusion to PD control derives from the fact that near a minimum of $\varphi$, (3) reduces to $u \approx -K_p(q - q^*) - K_d\dot{q}$, where $K_p = \frac{\partial^2\varphi}{\partial q^2}(q^*)$ is the Hessian of $\varphi$.

[2] Note that starting within $\mathcal{E}$ imposes a "speed limit" as well as a positional limit, since the total energy must be initially bounded [15].

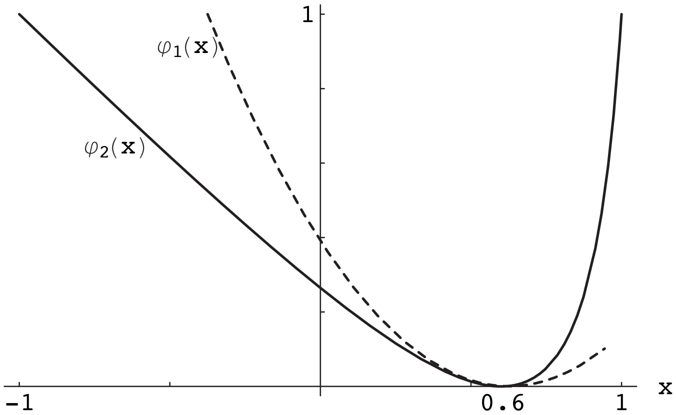


Fig. 1. An NF should evaluate uniformly at the boundary. In this case, $\mathcal{X} = [-1, 1]$, and $\varphi_1$ does not obtain the same value at $x = \pm 1$, whereas $\varphi_2$ does.

### F. Invariance under diffeomorphism

A key ingredient in the mix of geometry and dynamics involves the realization that NFs are invariant under diffeomorphism [2]. This affords the introduction of geometrically simple model spaces and correspondingly simple NFs.

### G. Why uniformly maximal at the boundary?

The requirement that an NF uniformly evaluates to a constant on the boundary is often overlooked. To illustrate this point, let $\mathcal{X} = [-1, 1]$, be the configuration manifold, and let $x^* = 0.6$ be the goal. The boundary, $\partial\mathcal{X}$, consists of two points, $-1$ and $+1$. As a candidate NF, one may naively consider

$$\varphi_1(x) = (x - 0.6)^2 \tag{5}$$

and note that $\nabla\varphi_1(x) = 2(x-0.6)$, which has no local minima on $[-1, 1]$ except at the goal of $x^* = 0.6$. If the system begins at the left edge of the configuration manifold, it will have a total initial potential of $1.6^2 = 2.56$, but the potential barrier on the right edge is only $0.16$. If the system is not highly over-damped, there are no guarantees that the right boundary will be avoided for all zero-velocity initial conditions within the domain $\mathcal{X}$. Thus, *second order safety* requires *boundary uniformity*. The NF

$$\varphi_2(x) = \frac{(x - 0.6)^2}{(1 - x^2) + (x - 0.6)^2}, \tag{6}$$

satisfies this requirement, as illustrated in Fig. 1.

## III. Composing NFs on Cross Product Spaces

### A. Construction

Consider two compact manifolds with corners, $\mathcal{X}_1$ and $\mathcal{X}_2$, together with associated NFs, $\varphi_1$ and $\varphi_2$. Denote the Cartesian product space, $\mathcal{Q} = \mathcal{X}_1 \times \mathcal{X}_2$. As a naive attempt to construct a navigation function on $\mathcal{Q}$, consider the sum of NFs, $\varphi : \mathcal{Q} \to [0, 1]$, given by

$$\varphi(q) = \tfrac{1}{2}(\varphi_1(x_1) + \varphi_2(x_2)),$$

where $q = (x_1, x_2)$. Such a function is Morse with unique minimum at $q^* = (x_1^*, x_2^*) \in \mathcal{Q}$, where $x_1^* \in \mathcal{X}_1$, $x_2^* \in \mathcal{X}_2$ are the respective goals of each NF. However, $\varphi$ is *not* an NF on $\mathcal{Q}$. The problem arises because $\varphi$ is not uniformly maximal on the boundary of $\mathcal{Q}$, given by the disjoint union of three components

$$\partial\mathcal{Q} = (\partial\mathcal{X}_1 \times \partial\mathcal{X}_2) \cup (\partial\mathcal{X}_1 \times \mathring{\mathcal{X}}_2) \cup (\mathring{\mathcal{X}}_1 \times \partial\mathcal{X}_2)\,.$$

For example, $\varphi(\cdot) < 1$ on $\partial\mathcal{X}_1 \times \mathring{\mathcal{X}}_2$, since $\varphi_2(\cdot) < 1$ on $\mathring{\mathcal{X}}_2$. Thus, while the sum of two NFs might be adequate for first-order kinematic systems using simple gradient descent, it does not ensure safety with respect to the boundary when lifted to the second-order dynamic setting as described in Sec. II-G.

Note that the function given by $q \mapsto (\varphi_1(x_1), \varphi_2(x_2))$, maps the boundary, $\partial\mathcal{Q}$, to the "top and right edges" of $[0, 1] \times [0, 1]$, namely $\{(1, \cdot)\} \cup \{(\cdot, 1)\}$ and encodes the goal $q^*$ at the point $(0, 0)$. This motivates the following definition.

*Definition 2:* A *composition function* is a functional $\vartheta : [0, 1] \times [0, 1] \to [0, 1]$ such that

1) $\vartheta(0, 0) = 0$;
2) $\vartheta(1, \cdot) = \vartheta(\cdot, 1) = 1$;
3) $\vartheta$ is $C^r$, $r \geq 2$, everywhere but at the point $(1, 1)$, where it is $C^0$.
4) $\vartheta$ is monotone increasing in both variables, i.e. $\frac{\partial\vartheta(z_1,z_2)}{\partial z_i} > 0$, $i = 1, 2$.

We are now ready to compose two NFs.

*Theorem 2 (Navigation Function Product):* Consider two compact manifolds with corners, $\mathcal{X}_1$ and $\mathcal{X}_2$, together with associated NFs, $\varphi_1$ and $\varphi_2$, and a composition function, $\vartheta$. The *navigation function product*

$$\varphi = \varphi_1 \vee \varphi_2,$$

given by $\varphi(q) = \vartheta(\varphi_1(x_1), \varphi_2(x_2))$, is a navigation function on $\mathcal{Q} = \mathcal{X}_1 \times \mathcal{X}_2$, with unique global minimum at $q^* = (x_1^*, x_2^*) \in \mathcal{Q}$.

*Proof: (Uniformly maximal)* Since $\vartheta$ is a composition function, $\varphi$ achieves the value of 1 exactly when either $\varphi_1$ and/or $\varphi_2$ achieve a value of 1, which comprises the entire boundary $\partial\mathcal{Q}$.

*(Smooth)* $\varphi$ is the composition of continuous functions and is therefore continuous ($C^0$) everywhere on $\mathcal{Q}$. Moreover, $\varphi^{-1}([0, 1)) = \mathring{\mathcal{Q}} = \mathring{\mathcal{X}}_1 \times \mathring{\mathcal{X}}_2$, and since $\varphi_i$ is $C^r$ on $\mathring{\mathcal{X}}_i$, $i = 1, 2$ and $\vartheta$ is $C^r$ on $[0, 1) \times [0, 1)$, then on $\mathring{\mathcal{Q}}$, $\varphi$ is the composition of $C^r$ functions, and is therefore $C^r$.

*(Morse)* Note that

$$\nabla\varphi = \begin{bmatrix} \frac{\partial\vartheta(z_1,z_2)}{\partial z_1}\nabla\varphi_1 \\ \frac{\partial\vartheta(z_1,z_2)}{\partial z_2}\nabla\varphi_2 \end{bmatrix}\Bigg|_{z_1=\varphi_1, z_2=\varphi_2}$$

which is well defined on $\mathring{\mathcal{Q}}$. By Definition 2, $\frac{\partial\vartheta(z_1,z_2)}{\partial z_i} > 0$, $i = 1, 2$, so on $\mathring{\mathcal{Q}}$ we have

$$\nabla\varphi(q) = 0 \iff \nabla\varphi_1(x_1) = 0 \text{ and } \nabla\varphi_2(x_2) = 0\,.$$

Thus, the critical points of $\varphi$ are given simply by all combinations of critical points of $\varphi_1$ and $\varphi_2$. The Hessian at a critical point[3] is given by

$$\frac{\partial^2\varphi}{\partial q^2} = \left.\begin{bmatrix} \frac{\partial\vartheta(z_1,z_2)}{\partial z_1}\frac{\partial^2\varphi_1}{\partial x_1^2} & 0 \\ 0 & \frac{\partial\vartheta(z_1,z_2)}{\partial z_2}\frac{\partial^2\varphi_2}{\partial x_2^2} \end{bmatrix}\right|_{z_1=\varphi_1, z_2=\varphi_2.} \tag{7}$$

Thus, the Hessian matrix evaluated at a critical point (7) is block diagonal with the positively scaled Hessians of each constituent NF on the diagonal. Thus, since the constituent Hessians are nondegenerate at a critical point, then $\varphi$ is also nondegenerate.

*(Unique minimum)* The critical point $q^* = (x_1^*, x_2^*)$ corresponding to a minimum of both constituent potential functions is also a minimum. Moreover, it is the global minimum since $\varphi(q) = 0$ iff $\varphi_i(x_i) = 0$, $i = 1, 2$, which is only true at $q^*$. By inspection of the Hessian, all other others critical points are saddles and maxima. ■

### *B. Designing composition functions*

The composition functions presented below, while by no means exhaustive, have certain convenient properties, such as associativity, ability to be tuned, and successive compositions reduce to a single, computationally simple expression. Consider the "squashing diffeomorphism" $\sigma : [0,\infty) \to [0, 1)$, defined by

$$\sigma(\alpha) := \frac{\alpha}{1+\alpha} \tag{8}$$

and the function, $\chi : [0, 1) \times [0, 1) \to [0,\infty)$, given by

$$\chi(z_1, z_2) = \frac{z_1}{1-z_1} + \frac{z_2}{1-z_2}. \tag{9}$$

Finally, define $\vartheta : [0, 1] \times [0, 1] \to [0, 1]$ by

$$\vartheta(z_1, z_2) := \begin{cases} \sigma \circ \chi(z_1, z_2), & \text{when } z_1 < 1 \text{ and } z_2 < 1 \\ 1, & \text{otherwise.} \end{cases} \tag{10}$$

To see that this function satisfies Definition 2 note that

$$\vartheta(z_1, z_2) = \frac{z_1 + z_2 - 2z_1z_2}{1 - z_1z_2} \tag{11}$$

everywhere on $[0, 1] \times [0, 1]$, except at $z_1 = z_2 = 1$ (i.e. the "upper right corner"). In particular, note that the above expression evaluates to 1 when either $z_1 = 1$ or $z_2 = 1$ (but the expression is not well defined for $z_1 = z_2 = 1$). Nevertheless, the limit

$$\lim_{z_1,z_2\to 1} \frac{z_1 + z_2 - 2z_1z_2}{1 - z_1z_2} = 1$$

and therefore $\vartheta$ is continuous, and well defined on its domain $[0, 1] \times [0, 1]$. Furthermore, the Jacobian of $\vartheta$,

$$\frac{\partial\vartheta(z_1, z_2)}{\partial z} = \frac{1}{(1 - z_1z_2)^2}\left[(1 - z_2)^2 \qquad (1 - z_1)^2\right],$$

is well defined except when $z_1 = 1$ or $z_2 = 1$, and can be easily extended by taking the limit as $z_1 \to 1$ or $z_2 \to 1$ (but not both) to yield

$$\lim_{z_1\to 1}\left(\frac{\partial\vartheta(z_1, z_2)}{\partial z}\right) = [1, 0], \ \lim_{z_2\to 1}\left(\frac{\partial\vartheta(z_1, z_2)}{\partial z}\right) = [0, 1].$$

[3]This expression is not valid away from the critical points, since it explicitly uses the fact that $\nabla\varphi_i = 0$, $i = 1, 2$.

As can be seen, the Jacobian is discontinuous at the point $z = (1, 1)$. It is also straight forward to compute the Hessian matrix, which is well defined everywhere on the domain $[0, 1] \times [0, 1]$ except at $z = (1, 1)$.

*Remark 1 (Semigroup property):* By Theorem 2, if $\varphi_1$ and $\varphi_2$ are, respectively, NFs on compact manifolds with boundary, $\mathcal{X}_1$ and $\mathcal{X}_2$, then $\varphi_1 \vee \varphi_2$ is an NF on $\mathcal{X}_1 \times \mathcal{X}_2$, and thus $\vee$ is closed. Moreover, Cartesian products of spaces are associative, i.e.

$$(\mathcal{X}_1 \times \mathcal{X}_2) \times \mathcal{X}_3 = \mathcal{X}_1 \times (\mathcal{X}_2 \times \mathcal{X}_3).$$

Finally, the navigation product is associative because $\vartheta$ is associative, namely $\vartheta(z_1, \vartheta(z_2, z_3)) = \vartheta(\vartheta(z_1, z_2), z_3)$. This can be verified by direct algebraic substitution. Thus, the set NFs on compact manifolds with boundary forms a semigroup. ■

*Remark 2 (Fold):* Let $\mathcal{X}_i$, $i = 1, \ldots, n$ be $n$ manifolds with boundary and let $\varphi_i$ be their respective NFs with goals $x_i$, $i = 1, \ldots, n$. Using the composition function $\vartheta$, let $\varphi = \varphi_1 \vee \varphi_2 \cdots \vee \varphi_n$, $\mathcal{Q} = \mathcal{X}_1 \times \mathcal{X}_2 \times \cdots \mathcal{X}_n$, and let $q = (x_1, x_2, \ldots, x_n) \in \mathcal{Q}$. Then

$$\varphi(q) = \sigma\left(\sum_{i=1}^{n} \frac{\varphi_i(x_i)}{1 - \varphi_i(x_i)}\right).$$

(Proof by induction). By definition, this is true for $n = 2$. Suppose it is true for $n = k$, and let $\varphi^k := \varphi_1 \vee \varphi_2 \cdots \vee \varphi_k$ be the navigation product of the first $k$ functions. Dropping the explicit dependence on the $x_i$'s, from (11) we have that

$$\varphi^{k+1} = \vartheta\big(\varphi^k, \varphi_{k+1}\big) = \frac{\varphi_{k+1} + \sigma\Big(\sum_{i=1}^k \frac{\varphi_i}{1-\varphi_i}\Big) - 2\varphi_{k+1}\sigma\Big(\sum_{i=1}^k \frac{\varphi_i}{1-\varphi_i}\Big)}{1 - \varphi_{k+1}\sigma\Big(\sum_{i=1}^k \frac{\varphi_i}{1-\varphi_i}\Big)}$$

which, upon multiplying numerator and denominator by $1 + \sum_{i=1}^k \frac{\varphi_i}{1-\varphi_i}$, and simplifying yields the desired result. ■

*Tunable composition function.* While the NF framework affords certain formal guarantees, such as dynamical obstacle avoidance and essentially global convergence, one of the practical limitations of the existing NF literature concerns tuning NF-based controllers. In (3), the damping gain, $K_d$, comprises a set of a free design parameters, but there are no explicit hooks for tuning the potential function, unless the designer builds such hooks directly into the NF (as done, for example, in [16]). The composition function formed by replacing $\chi$ (9) with

$$\chi_k(z_1, z_2) = k_1\frac{z_1}{1-z_1} + k_2\frac{z_2}{1-z_2}$$

introduces a set of tunable gains $k = (k_1, k_2)$. As long as $k_1, k_2 > 0$, it is readily shown that the resulting composition function

$$\vartheta_{\mathrm{k}}(z_1, z_2) := \begin{cases} \sigma \circ \chi_k(z_1, z_2), & \text{when } z_1 < 1 \text{ and } z_2 < 1 \\ 1, & \text{otherwise.} \end{cases} \tag{12}$$

satisfies Definition 2.

With appropriate choice of gains at each stage of composition, the $\vee$ is associative as before. For example

$$\vartheta_{(k_1,1)}\big(z_1, \vartheta_{(k_2,k_3)}(z_2, z_3)\big) = \vartheta_{(1,k_3)}\big(\vartheta_{(k_1,k_2)}(z_1, z_2), z_3\big),$$

and $\varphi = \varphi_1 \vee \varphi_2 \cdots \vee \varphi_n$ is given by

$$\varphi(q) = \sigma\left(\sum_{i=1}^{n} k_i \frac{\varphi_i(x_i)}{1-\varphi_i(x_i)}\right).$$

## IV. Simple Illustrative Examples

### A. Cross product of a circle and an interval

Note that $\varphi_1(\theta) = (1-\cos\theta)/3$ is an NF on $\mathrm{S}^1$ with a goal at $\theta = 0$. The maximum value of $\varphi_1$ is chosen to be $2/3$, rather 1, since there is no boundary on $\mathrm{S}^1$. The function $\varphi_2(x) = x^2$ is an NF on $[-1,1]$, with a goal at $x = 0$. Let $\mathcal{Q} = \mathrm{S}^1 \times [-1,1]$, and let $q = (\theta, x) \in \mathcal{Q}$. The navigation product $\varphi = \varphi_1 \vee \varphi_2 : \mathcal{Q} \to [0,1]$ is given by

$$\varphi(q) = (\varphi_1 \vee \varphi_2)(\theta, x) = \frac{1 + x^2 + \left(-1 + 2\,x^2\right)\cos\theta}{3 - x^2 + x^2\cos\theta}.$$

A 3D plot is shown in Figure 2.

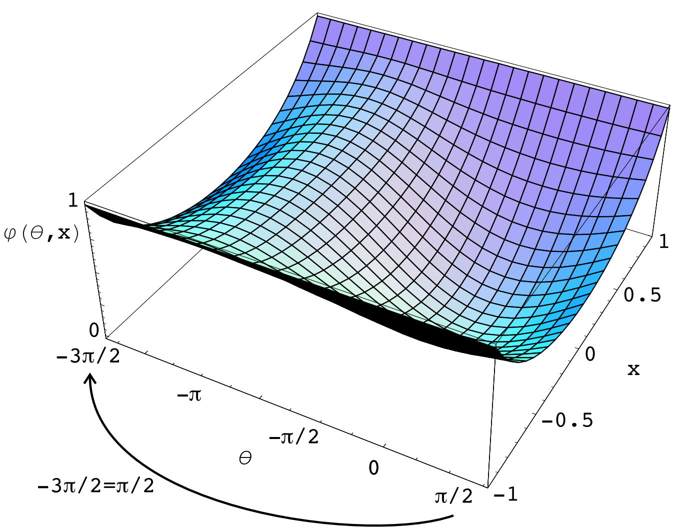


Fig. 2. A 3D plot of an NF on $\mathrm{S}^1 \times [-1,1]$ derived from two simple NFs on $\mathrm{S}^1$ and $[-1,1]$, respectively. The endpoints of the $\theta$ axis are identified. There are two critical points, a saddle and a minimum.

### B. Cross product of intervals

Let $\mathcal{X}_i = [\alpha_i, \beta_i]$, for some $\alpha_i, \beta_i \in \mathbb{R}$, be two intervals and consider a different NF, $\varphi_i : \mathcal{X}_i \to [0,1]$, for each

$$\varphi_i(x) = \frac{(x - q_i^*)^2}{(\beta_i - x)(x - \alpha_i) + (x - q_i^*)^2}, \quad i = 1, 2, \tag{13}$$

with goals at $q_i^*$, i=1,2. Let $\mathcal{Q} = \mathcal{X}_1 \times \mathcal{X}_2$, and let $q = (q_1, q_2) \in \mathcal{Q}$. As shown by the next example in which the configuration space is the cross product of intervals, the naive NF candidate given by

$$\varphi_{\text{naive}}(y) = \tfrac{1}{2}(\varphi_1(q_1) + \varphi_1(q_2))$$

is not admissible, which leads to collisions with the boundary.

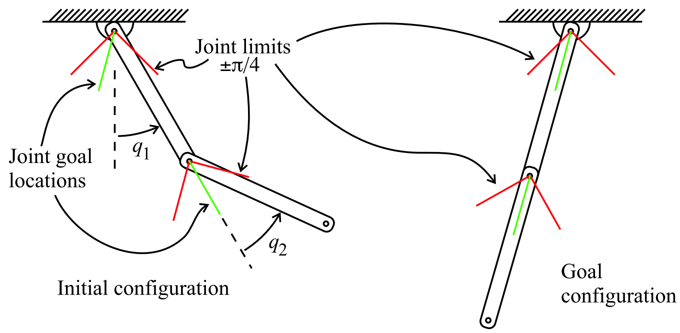


Fig. 3. A "double pendulum."

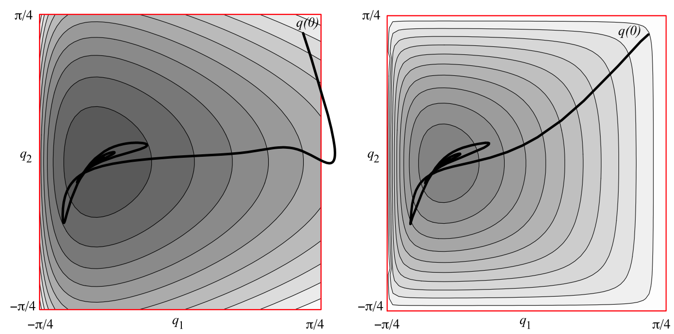


Fig. 4. Contour plots of two candidate NFs on the space $\mathcal{Q} = [-\frac{\pi}{4}, \frac{\pi}{4}]^2$. The configuration space trajectories resulting from a zero velocity initial condition, $(q(0), 0) \in T\mathcal{Q}$, of the double pendulum system shown in Fig. 3, subject to the "gradient + damping" feedback in (3) are shown by the bold curves. *Left.* The potential function, $\varphi_{\text{naive}}(q_1, q_2) = \frac{1}{2}(\varphi_1(q_1) + \varphi_2(q_2))$, is not an NF, because it is not uniformly maximal on the boundary, and thus the trajectory crosses the boundary. *Right.* The navigation product, $\varphi = \varphi_1 \vee \varphi_2$, from Theorem 2 is an NF, thus ensuring safety with respect to the boundary. Note that the closed interior of each level curve shown is a set $\mathcal{Q}_c$ as defined in II-D.

### C. Double pendulum

A simulated two link revolute-revolute mechanical system, as shown in Fig. 3, illustrates the consequences of naive artificial potential function design. Joint limits were set to $\pm\frac{\pi}{4}$, and thus $\mathcal{Q} = [-\frac{\pi}{4}, \frac{\pi}{4}]^2$. Simulated link lengths were $\ell_1 = \ell_2 = 1$ and masses of $m_1 = m_2 = 1$ located at the end of each respective link. Finally, the damping matrix was set to $K_d = \mathrm{diag}\{1.5, 0.5\}$.

For each DOF, the NF, $\varphi_i$, $i = 1, 2$, was given by (13), where the goals were chosen to be $(q_1^*, q_2^*) = (-\frac{3\pi}{20}, 0)$. The control law was given by (3) for both the "naive" potential function, $\varphi_{\text{naive}} = \frac{1}{2}(\varphi_1 + \varphi_2)$, and a true NF given by $\varphi = \varphi_1 \vee \varphi_2$.

If the links were mechanically uncoupled, then control based on, $\varphi_{\text{naive}}$, would guarantee that all zero-velocity initial conditions within the free configuration space, $\mathcal{Q}$, safely converge. However, the mechanical coupling between the links renders the behavior undesirable since the first link violates its joint limit, as shown in Fig. 4.

## V. Conclusions

The goal of this note is to enable a control-system designer to construct an obstacle avoiding "spring law" for each separate configuration component of a Lagrangian system, and then stitch the spring laws together in such a fashion that dynamical obstacle avoidance is maintained even for coupled mechanical systems. The proposed composition is associative, tunable, and computationally simple. Of course, designing the constituent navigation functions for each component may be challenging!

Building on Koditschek's observation that NFs exists for all manifolds with (smooth) boundary, the results of this note imply that any manifold comprised of the Cartesian product of multiple manifolds with boundary also admits an NF, despite the introduction of corners. The next step might be to show the existence of NFs on general manifolds with corners. Considerably more challenging would be to show the existence of NFs on Whitney stratified manifolds. Fortunately, manifolds with corners were sufficiently general to solve the problem at hand, namely "second order" navigation on Cartesian product manifolds, and Morse theory for manifolds with corners [12], [13] is considerably simpler than stratified Morse theory [17], [18].

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