An Almost Global Estimator on $\mathrm{SO}(3)$ with Measurement on S^2

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***Abstract*— This paper presents an almost globally convergent state estimator for the orientation of a rotating rigid body. The estimator requires knowledge of the angular velocity of the body at each time instant and the measurement consists of a single unit vector on the body, which we take without loss of generality to be the first column of the rotation matrix. The stability proof involves a relatively simple Lyapunov and invariance-like analysis. A mild non-degeneracy constraint on the control input guarantees the fulfillment of the invariance criterion. We apply the result to needle-tip orientation estimation for tip-steerable needles.**

I. INTRODUCTION

Many developments in control theory over the past 50 years have been driven by the desire for better attitude determination and control. Much of this work focuses on satellite attitude regulation, but estimation and control on the rotation group applies to a wide range of problems. Our particular treatment of the attitude estimation problem is motivated by a medical intervention, control of tip-steerable needles [1], [2]. The heart of our problem lies in estimating all three degrees of freedom of rotation given only a two degree-of-freedom measurement in the form of a unit vector—in our case, this unit vector corresponds to one of the columns of the rotation matrix. The goal is to determine the entire rotation matrix asymptotically, given that measurement plus knowledge of the control input.

This application drives the need for an estimator on $\mathrm{SO}(3)$ from measurement restricted to S^2, a result that could apply to any number of other fields involving attitude estimation for rotating rigid bodies. For example, attitude estimation in an underwater vehicle based on a gravity sensor or compass heading comes to mind. In this paper, we demonstrate an essentially global estimator for this problem; convergence only fails for initial estimates that are π radians away from the actual state—a set of measure zero. The proof of convergence is given using (local) exponential coordinates that are valid over the entire domain of attraction. The method extends easily to the measurement of multiple vectors, for which the convergence proof becomes trivial. Our estimator is invariant in the sense that it is described by a matrix differential equation with the property that when the initial condition is on the manifold described by the Lie group $\mathrm{SO}(3)$, exactly integrating the matrix differential equation will result in the estimate remaining in $\mathrm{SO}(3)$ for all time.

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A. Related work

Since the advent of the Kalman filter—the optimal *linear* filter—five decades ago, the standard approach for nonlinear systems has been to wrangle the system into a form amenable to applications of the Kalman filter, through coordinatization and linearization. As most problems are ultimately inherently nonlinear, this technique has been the practical, albeit suboptimal, workhorse of the state estimation systems for spacecraft, aircraft, submarines, and many other noisy nonlinear system. However, the method is fraught with pitfalls of divergence due to initial conditions and singularities in representation and statistics [3], [4].

In the search for an optimal invariant filter with robustness to initial error, it appears the field is taking the opposite route to Kalman and Luenberger, where the stochastic and deterministic cases were treated in that order in 1961 and 1964, respectively. Recent work by Bonnabel, Mahony, and others has begun to develop a principled framework for defining convergent, invariant observers [5]–[8]; much of this recent development takes advantage of the Lie group structure of $\mathrm{SO}(3)$.

Markley [9] attempted to develop an invariant filter, but the tools to solve the Fokker-Planck equations exactly were not available and many simplifying assumptions were made, rendering the solution suboptimal in a similar sense to an extended Kalman filter. The natural next step is to develop convergent invariant filters evolving from these invariant observers, as Kalman filters are to Luenberger observers. Recent work in stochastic processes on matrix Lie groups by Park and Chirikjian [10], [11] may provide the tools necessary to analytically solve (or at least approximate) the Fokker-Planck equations, where it was previously not possible. More recently work by Bonnabel [12] implements an invariant extended Kalman filter (IEKF) using their invariant observer structure to propagate state and using a linearization of the system for propagating statistics, which is optimal in the same sense that the EKF is approximately optimal for non-invariant observer representations of non-linear systems.

Recently Kinsey and Whitcomb developed invariant adaptive identifier methods for systems in which the (unknown) attitude was fixed and the inputs and the outputs of the system were known [13]. The work presented in this paper is similar in the choice of output error, except we allow the attitude of the system to be time varying. Another recent application was invariant estimation of the homography between stereo cameras [14]. One can imagine that the techniques could also be applied to the many applications in physics described by finite dimensional matrix Lie groups [15].

For our specific application of needle steering, previous work has been done to develop controller and estimator pairs in a reduced set of local coordinates [16]–[18]. These methods were used for both state estimation and control to a surface defined as part of the state reduction.

The key contribution of our estimator is to provide a nonlinear output injection term into the tangent space of the estimator configuration space—a copy of SO(3)—and to cast the resulting error dynamics as a nonautonomous matrix differential equation. Given this differential equation, we provide an associated proof of almost global convergence of the estimator error to the identity matrix. The nonlinear output injection is a function of the measurement of a single vector between the center of rotation and a point on the rotating body. To the best of our knowledge, the requirement of only one measurement vector is a less restrictive measurement model than all other invariant estimator work to date. This includes the most recent and complete work in this field about which we are aware [8] which requires multiple measurements or full state access. Of course, at present our method does not constitute a filter, much less an optimal filter; the previous discussion of attitude filtering defines our eventual goal for both needle steering and attitude determination in general.

B. *Paper Organization*

Section II describes the motivating example of needle steering and the generic plant and measurement model for any rotating rigid body; it also provides a general definition of invariant estimation on the group of rigid body rotations. In Section III, we prove that the system is observable for our plant and measurement model. Section IV defines our contribution of a nonlinear output injection term for the invariant estimator and provides the associated analysis of almost-global asymptotic stability. In Section V we present a few illustrative simulations of the method, for both needle steering and an arbitrary rotating rigid body. We conclude with a few observations and proposed directions for further research.

II. Motivation and Plant Model

The principal application that motivates the present study is the problem of estimating the orientation of a flexible, tip-steerable needle [1], [2], as depicted in Figure 1(A). As these needles are inserted into tissue, the tip asymmetry causes the needle to deflect and follow a circular arc. Rotation of the needle shaft outside the patient causes the needle to act as a "flexible drive shaft", reorienting the asymmetric tip before subsequent insertions.

Needle tip motion can be described as a left-invariant kinematic vector field on SE(3) [2]. For planning and control, one would ideally like an estimate of the position and orientation of the body-fixed frame at the tip of the needle. Unfortunately, current medical imaging modalities such as bi-plane fluoroscopy can only be used to extract five degrees of freedom of the needle: the position in $\mathbb{R}^3$ of the needle tip and the vector direction of the needle in S^2, but the tip orientation about the needle shaft cannot presently be resolved. So, for the present paper, we assume that we can measure the vector tangent to the needle shaft at the tip (namely, aligned with the x-axis in Figure 1(A)) either through external imaging or a magnetic tracking device.

A. *Motivation: Needle Steering*

Because the needle tip position can be directly measured, in this work we neglect the positional estimation problem, and focus our efforts on the more challenging attitude estimation problem. The orientation dynamics of the simplest needle steering kinematic model, called the "unicycle model", developed by Webster et al. [2] can be expressed as follows:

$$(R^{-1}\dot{R})^{\vee} = u(t) = \begin{bmatrix} \omega(t) \\ 0 \\ -\kappa v(t) \end{bmatrix}, \quad y = Re_1 \qquad (1)$$

where κ is the instantaneous curvature about the z-axis of the needle path during insertion, ω is the needle shaft rotational velocity, and v is the needle insertion speed. Here, y represents the measurement of the orientation of the x axis. The operators $\hat{}: \mathbb{R}^3 \to \mathfrak{so}(3)$ and $^{\vee}: \mathfrak{so}(3) \to \mathbb{R}^3$ denote the usual isomorphism between $\mathbb{R}^3$ and $\mathfrak{so}(3)$, the Lie algebra of SO(3).

The practical question addressed by this paper is how to estimate $R(t) \in \mathrm{SO}(3)$ asymptotically based on the measurement $y(t) \in \mathrm{S}^2$.

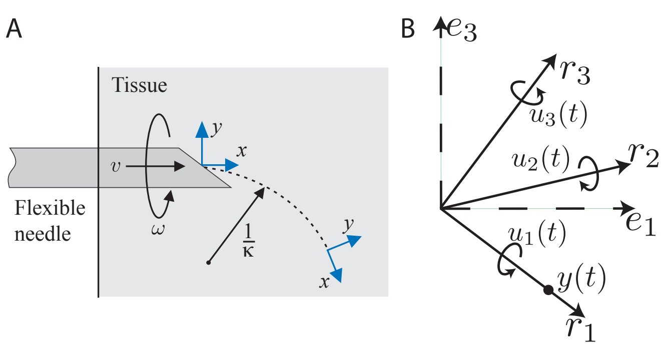

Fig. 1. Kinematic models for a rotating rigid body. (A) Kinematic model used to represent the insertion of tip-steerable flexible needles [2]. As the needle is inserted at velocity v, a bevel or other tip asymmetry causes a lateral force that deflects the needle along a curved path of radius $1/\kappa$. Rotations about the base of angular velocity ω reorient the bevel tip (Modified with permission from [11]). (B) General rotating rigid body model (that includes the needle kinematic model). The body frame velocities, $u_i \in \mathbb{R}$, about each rotation axis $r_i \in \mathbb{R}^3$, $i = 1, 2, 3$, are known. The unit vectors e_i, $i = 1, 2, 3$, represent the world frame. The measurement (or output model) is a single vector, $y(t)$, defined by a point on the rotating rigid body with respect to the center of rotation; as depicted, we assume (without loss of generality) that $y(t) = Re_1 = r_1$.

B. *Plant Model and Measurement Model*

As a slight generalization of the needle steering orientation estimation problem, we consider a kinematic rotating rigid body in which the angular velocity with respect to the body fixed frame is known; see Figure 1(B). As depicted, we also assume that we can measure a single point a unit distance from the center of rotation of the rigid body. For sake of notational simplicity and without loss of generality, we assume that this point is aligned with the first axis of the rigid body.

Consider the left invariant kinematic system described using the Lie group $\mathrm{SO}(3)$ and its corresponding Lie algebra $\mathfrak{so}(3)$,

$$R^T \dot{R} = \hat{u}, \tag{2}$$

with output map

$$y = Re_1. \tag{3}$$

We assume that we know the body frame velocities of the rigid body, u.

C. Problem Statement

We wish to define an invariant estimator for $R(t) \in \mathrm{SO}(3)$ for the system defined by (2)–(3). The estimator should be invariant in the sense that the estimate $\widetilde{R}$ should evolve on the manifold $\mathrm{SO}(3)$. We seek an estimatator that converges asymptotically to the true value, $R(t)$, as $t \to \infty$ from essentially any initial condition in $\mathrm{SO}(3)$. As will be seen, this will require us to put some constraints on the control inputs, $u(t)$.

Our estimator structure is defined on $\mathrm{SO}(3)$ by

$$\tilde{R}^T \dot{\tilde{R}} = \hat{u} + g(y, \tilde{y}), \tag{4}$$

where the estimator output is

$$\tilde{y} = \tilde{R} e_1, \tag{5}$$

and $g(y, \tilde{y}) \in \mathfrak{so}(3)$.

III. Observability of a Kinematic Rigid Body Based on a Single Vector Measurement

Before presenting our observer formulation, we examine the observability of the dynamics given by (2)–(3).

A. Observability

A first step to solving the estimation problem is to ascertain whether the system is observable with the measurement of a single point on the rigid body. One notion of observability is whether the state of the system can be determined from the output of the system and its $n-1$ derivatives [19]. Our approach follows Kallem et al. [18] for slightly different plant and output models.

Lemma III.1. *The system described in* (2) *and* (3) *is observable at time t provided the input $u(t) \neq \alpha e_1$ is known.*

Proof. The output and its first time derivative are

$$y = Re_1 = \mathbf{r}_1 \tag{6}$$

and

$$\dot{y} = \dot{R} e_1 = R R^T \dot{R} e_1 = R \hat{u} e_1 = R \begin{bmatrix} 0 \\ u_3 \\ -u_2 \end{bmatrix}. \tag{7}$$

Manipulating these expressions, one can show that

$$\mathbf{r}_2 = \underbrace{\frac{u_2}{u_2^2 + u_3^2}}_{\alpha} (y \times \dot{y}) + \underbrace{\frac{u_3}{u_2^2 + u_3^2}}_{\beta} \dot{y}, \tag{8}$$

and thus

$$\mathbf{r}_3 = \mathbf{r}_1 \times \mathbf{r}_2 = \alpha \left[y \times (y \times \dot{y}) \right] + \beta (y \times \dot{y}). \tag{9}$$

We see that if the restriction placed on the control input is violated, then the computation of (8) becomes ill defined. □

Note that observability of (2)–(3) requires that the control input not correspond to a pure rotation about the output vector, i.e. $u \neq \alpha e_1$. This implies u_2 and u_3 cannot both be zero—such a rotation renders $y = \mathrm{const.}$, making it impossible to infer the final degree of freedom.

IV. Invariant Observer for Attitude Estimation

In this section, we develop an estimator for the attitude of the rigid body that is almost globally convergent, with the exception of a set of initial conditions of measure zero. Many techniques for attitude estimation employ embeddings or coordinatizations (in the context of needle steering, see [18]) of the rotation matrix describing the attitude of the rigid body. Our estimator consists of a matrix differential equation evolving on the manifold $\mathrm{SO}(3)$, such that the estimator state remains on the manifold for all time.

Given the representation of the kinematic rotating rigid body (2)-(3), we propose the following as the estimator correction term for the estimator (4)-(5):

$$g(y, \tilde{y}) = k \tilde{R}^T \widehat{(y \times \tilde{y})} \tilde{R}. \tag{10}$$

Note that $g(y, \tilde{y})$ lies in $\mathfrak{so}(3)$, ensuring that the estimator evolves on $\mathrm{SO}(3)$.

In the absence of a control input, the output injection term (10) would cause a rotation of the estimate about an axis mutually orthogonal to both the first column of the estimator, $\widetilde{r}_1$, and that of the rigid body, r_1, thereby aligning these two columns asymptotically. Thus the estimator would remain in error by some rotation about e_1, i.e. $\widetilde{R} = \exp\{\alpha \hat{e}_1\} R$. Fortunately, when this is the case, "most" inputs to the system drive $\widetilde{r}_1$ and r_1 apart, which is at the heart of our invariance-like stability analysis, as shown below.

A. Stability Analysis

We define the error between the kinematic system and the observer as

$$E = R^T \tilde{R}. \tag{11}$$

For a convergent estimator, this error will approach the identity matrix as time approaches infinity. By direct computation,

$$E^T \dot{E} = \hat{u} - E^T \hat{u} E + k \tilde{R}^T \left(\widehat{y \times \tilde{y}} \right) \tilde{R}. \tag{12}$$

Since

$$y \times \tilde{y} = (Re_1) \times (\tilde{R} e_1) = R \hat{e}_1 E e_1, \tag{13}$$

the body-frame error velocities are given by

$$\Omega_E = (E^T \dot{E})^\vee = u - E^T u - k\hat{e}_1 E^T e_1. \quad (14)$$

Before proceeding to develop a candidate Lyapunov function, we point out an important property of the velocities of error system (14): they have no explicit dependence on the actual state of the system, $R(t), \widetilde{R}$, but rather only depend on the error between them. While this is a standard result for the LTI Luenberger observer, this is not generally true of nonlinear systems; for other choices of nonlinear output injection than (10), the body frame error will typically depend on both R and $\widetilde{R}$.

Now, we represent the error system in terms of exponential coordinates,

$$E = \exp(\hat{x}), \quad (15)$$

which serve as local coordinates for stability analysis of estimator error. Note that our choice of coordinates cover all of SO(3), excluding a thin set of rotations, namely rotations about any axis of π radians.

The Jacobian between exponential coordinates and body-frame velocities [20] allows us to recast (14) in terms of exponential coordinates via $\dot{x} = J_R^{-1}(x)\Omega_E$, namely:

$$\begin{aligned} \dot{x} &= J_R^{-1}(x) \left[u(t) - \exp(\hat{x}^T)u(t) - k\hat{e}_1 \exp(\hat{x}^T)e_1\right] \\ &= f(x,t). \end{aligned} \quad (16)$$

Equation (16) represents the flows of the error system in terms of a typical nonlinear, nonautonomous differential equation.

Consider the following candidate Lyapunov function on the local coordinates of the error system,

$$V = \frac{1}{2}x^T x, \quad (17)$$

with time derivative

$$\begin{aligned} \dot{V} &= x^T \dot{x} \\ &= x^T J_R^{-1}(x)\Omega_E \\ &= x^T \Omega_E \\ &= x^T \Big(\underbrace{u - E^T u}_{A} + k \underbrace{\widehat{E^T e_1} e_1}_{B} \Big). \end{aligned} \quad (18)$$

The penultimate step of (18) is due to the fact that x is a left eigenvector of $J_R^{-1}(x)$ associated with a unity eigenvalue. We are now left to show that the time derivative of the Lyapunov function implies asymptotic stability. The following computations show that $\dot{V}$ is negative semi-definite for the chosen Lyapunov function. This is accomplished by expanding the error state, E, in (18) using the Rodrigues formulation for exponential coordinates given in (15) and writing x in axis and angle form:

$$\phi = \|x\|, \quad \hat{q} = \frac{\hat{x}}{\|x\|}. \quad (19)$$

Then, A from (18) can be simplified as

$$\begin{aligned} A &= u - \left(I - \hat{q}\sin(\phi) + \hat{q}^2(1-\cos(\phi))\right) u \\ &= -\sin(\phi)\hat{u}q - (1-\cos(\phi))\hat{q}^2 u. \end{aligned} \quad (20)$$

A similar computation yields B from (18):

$$B = -\sin(\phi)(e_1 e_1^T - I)q - (1-\cos(\phi))\left[\hat{e}_1 q q^T e_1\right]. \quad (21)$$

Thus, the body frame error velocity (14) is

$$\begin{aligned} \Omega_E &= A + kB \\ &= -\sin(\phi)\hat{u}q - (1-\cos(\phi))\hat{q}^2 u + \\ &\quad k\left(-\sin(\phi)(e_1 e_1^T - I)q - (1-\cos(\phi))\left[\hat{e}_1 q q^T e_1\right]\right). \end{aligned} \quad (22)$$

Using (20) and (21),

$$\begin{aligned} \dot{V} =& x^T (A + kB) \\ =& \phi q^T (-\sin(\phi)\hat{u}q - (1-\cos(\phi))\hat{q}^2 u \\ & - k\sin(\phi)(e_1 e_1^T - I)q \\ & - k(1-\cos(\phi))\left[\hat{e}_1 q q^T e_1\right]) \\ =& -k\phi\sin(\phi)q^T(e_1 e_1^T - I)q \\ =& x^T \underbrace{\left(-k\,\mathrm{sinc}(\phi)\hat{e}_1^2\right)}_{M(\phi)} x \\ \leq& x^T \underbrace{\left(-k\,\mathrm{sinc}(\phi(t_0))\hat{e}_1^2\right)}_{M(\phi(t_0))} x \\ =& V^*(x) \leq 0 \quad (\forall k < 0). \end{aligned} \quad (23)$$

Because $\mathrm{sinc}(\phi) > 0$ on $\phi \in [0, \pi)$ and $\hat{e}_1^2$ is negative semidefinite; thus for $k < 0$ then $\dot{V} \leq 0$. We see that the time derivative of the chosen Lyapunov function is only negative semi-definite in the parameterized error coordinates and Lyapunov analysis only guarantees asymptotic convergence to the set $\{x : V^*(x) = 0\} = \{x : x = \alpha e_1, \alpha \neq 0\}$, or, equivalently, the set $\{x : \dot{V} = 0\}$.

As a short aside, we earlier discussed the choice of using e_1 in the output map to clarify the presentation. Here, we point out that had we chosen a different vector in the output map, say $y(t) = R(t)v^*$, then the matrix $M(\phi)$ in (23) would still be a rank 2 matrix with v^* lying in the null space, and the points to which Lyapunov analysis guarantees asymptotic convergence are the scalar multiples of v^*. When multiple measurements are available, the output injection term in (10) can be written as a sum of the cross products of the outputs and the result is that $M(\phi)$ from (23) is strictly negative definite. Thus for the multiple measurement case, traditional Lyapunov theory guarantees asymptotic convergence, without resorting to the invariance-like methods used subsequently in the single measurement case.

To show asymptotic convergence to $x = 0$, we now must consider whether the control input renders the system, in global and local coordinates given in (12) and (16) respectively, autonomous or nonautonomous. In the case of a control input that is constant or a function of state, the system is autonomous and we can proceed with a straightforward proof using LaSalle's invariance principle and the theorems of Barbashin or Krasovskii. For an arbitrary control input, an explicit function of time, we resort to one of the techniques called "invariance-like" methods by Khalil [21]. In particular, we use the theorem first presented by Matrosov [22] and extended to non-scalar auxiliary functions by Rouche [23]. We refer readers to the translated Matrosov paper and the Rouche paper for proofs of the original theorems.

1) The vector-field on the set $\mathcal{M} = \{x : \dot{V} = 0\}$: From (14) note that when $q = e_1$ (i.e. $x = \alpha e_1$) we have

$$\Omega_E = [I - E^T]u, \tag{24}$$

where $E = \exp(\phi \hat{e}_1)$. Thus, by direct computation

$$\dot{x} = J_R^{-1}(\phi e_1)\Omega_E = \begin{bmatrix} 0 \\ -\phi u_3 \\ \phi u_2 \end{bmatrix}. \tag{25}$$

This shows that the non-invariance of $\mathcal{M}$ depends on the control input $u(t)$. If we assume that $u(t)$ is bounded and piecewise continuous in time, then it is easy to show that $f(x,t)$ is differentiable in space and time and thus locally Lipschitz on $\{x \in \mathbb{R}^3 : \|x\| < \pi\}$. Further, we impose a "persistency" condition on $u(t)$, namely that u_2 and u_3 are not both zero simultaneously when the system state is on the problem set $\mathcal{M}$. Written more formally, let

$$u_2^2(t) + u_3^2(t) > \delta^2 > 0 \text{ for } x \in \mathcal{M}. \tag{26}$$

For this input, it is easy to see from (25) that the vector field is transverse to $\mathcal{M}$ instantaneously when on the problem set. We wish to show the vector field is transverse over some nonzero interval of time and hence there exists a time $t_1 > t$ such that $x(t_1) \notin \mathcal{M}$. The existence of a finite time for which $x(t)$ exits an open region containing the problem set lies at the core of the theorems of Matrosov and Rouche, which can be viewed as analogues to LaSalle's invariance principle for non-autonomous systems.

2) Main Result: Because the control input, $u(t)$, is not necessarily constant, this system is generally nonautonomous and thus the traditional theorems concerning LaSalle's invariance principle are not valid. This is due to the fact that for general nonautonomous systems, the positive limit sets of solutions to the differential equations are not invariant [21].

However, we have a system in which the Lyapunov function and its time derivative are not explicit functions of time, despite the fact that the system is nonautonomous. This simplifying feature of our problem is a direct result of the choice of nonlinear output injection and the resulting independence of the error system (14) from the state, R, of the system (2). This feature makes the application of the aforementioned invariance-like method more simple to show convergence. Here we restate the theorem as given by Rouche in the notation of this paper, without proof, for reader convenience.

Definition 1 (Non-vanishing-definite vector function)**.** *The vector function $Y(t,x) : \mathbb{R} \times B_\rho \to \mathbb{R}^k$, where k is a positive integer, is **non-vanishing definite on a set** $\mathcal{M}$ if, for every pair of positive numbers υ and ε with $\upsilon < \varepsilon < \rho$, there is a positive number $\xi(\upsilon, \varepsilon)$ and an open covering $\{\pi_1, \pi_2, \cdots, \pi_m\}$ of the set*

$$F = \{x : \upsilon \leq \|x\| \leq \varepsilon, d(x, \mathcal{M}) = 0\}$$

such that for every i, $(1 \leq i \leq m)$, there is a component Y_j of Y with the property that $(t \in \mathbb{R})(x \in \pi_i) \Rightarrow |Y_j(t,x)| > \xi$.

Theorem IV.1 (Rouche Theorem 4.4 from [23]: sufficient conditions for uniform asymptotic stability)**.** *Let there exist two functions $V(t,x) : \mathbb{R} \times B_{\rho'} \to \mathbb{R}$ and $W(t,x) : \mathbb{R} \times B_{\rho'} \to \mathbb{R}^k$ (k a positive integer), continuous as well as their time derivative $\dot{V}(t,x)$ and $\dot{W}(t,x)$ computed along the solutions of $\dot{x} = f(t,x)$. If*

- *(a) for all $x \in B_\rho : \|f(t,x)\| \leq A$, where A is a positive constant;*
- *(b) $V(t,x)$ is positive definite; $V(t,x) \to 0$ uniformly in t when $x \to 0$;*
- *(c) there exists a continuous function $V^*(x) : B_{\rho'} \to \mathbb{R}$ such that $\dot{V}(t,x) \leq V^*(x) \leq 0$; $\dot{V}(t,0) = 0$; we write $\mathcal{M} = \{x : V^*(x) = 0\}$;*
- *(d) for every $L > 0$, there is a $\chi > 0$ such that*

$$d(x, \mathcal{M}) \leq \chi \Rightarrow \|W(t,x)\| < L$$

- *(e) $\dot{W}(t,x)$ is non-vanishing definite on $\mathcal{M}$;*

the, the vanishing solution $x \equiv 0$ of the system $\dot{x} = f(t,x)$ is uniformly asymptotically stable.

Using this theorem, we develop a corollary for our error system kinematics.

Proposition IV.2 (Corollary to Theorem IV.1)**.** *Consider the nonautonomous system represented in local coordinates as given in* (16)*, with a Lyapunov function and its negative semidefinite time derivative given by* (17) *and* (23)*, respectively. Then, if the control input $u(t)$ is bounded, continuous, and persistently exciting in the sense of* (26)*, the origin of the error system, $x = 0$ is asymptotically stable.*

Proof. Let

$$W(t,x) = \hat{e}_1^2 x \tag{27}$$

with the open balls B_ρ and $B_{\rho'}$ defined with $\rho' = \pi$, $\rho = \pi - \epsilon$, and $\epsilon > 0$.

- (a) For bounded control input $u(t)$ and bounded estimator gain k, the body frame velocity Ω_E is bounded. In addition, the Jacobian $J_R^{-1}(x)$ is nonsingular for all $x \in B_\rho$ such that $\exists A > 0$ where the time derivative of the error coordinates are bounded by A, $\|\dot{x}\| = \|f(t,x)\| < A$;
- (b) $V(t,x) = \frac{1}{2}x^T x > 0, V(t,0) = 0$ by construction;
- (c) Let $M(\rho) = -k\text{sinc}(\rho)\hat{e}_1^2$ identified with the semidefinite function $M(\cdot)$ given in (18). Then the time derivative of the Lyapunov equation is bounded from above as $\dot{V}(t,x) \leq x^T M(\rho) x = V^*(x) \leq 0$ (where $M(\rho)$ is negative semidefinite, as shown in (23)), $\dot{V}(t,0) = 0$, $\forall t > t_0$, and the set where $\dot{V}(t,x) = 0$ is $\mathcal{M} = \{x : x = \alpha e_1\}$;
- (d) The distance of the current state from the problem set $\mathcal{M}$ is $d(x,F) = x_2^2 + x_3^2$. The magnitude of the vector valued auxiliary function $W(t,x)$ is $\|W(t,x)\| = x_2^2 + x_3^2$. So for every $L > 0$, let $\chi = \frac{L}{2}$ such that $d(x,\mathcal{M}) \leq \frac{L}{2} \Rightarrow \|W(t,x)\| < L$;

(e) Recall from (19) that $x = \phi q$, where $||q|| = 1$, and note that on the problem set $\mathcal{M}$ the magnitude of the angle error, ϕ, is given by $\phi^2 = ||x||^2 = x_1^2$, and $q = \pm e_1$. Thus the time derivative of the auxiliary function $W(t,x)$ is

$$\dot{W}(t,x) = \hat{e}_1^2\dot{x} = \hat{e}_1^2 J_R^{-1}(\phi q)\Omega_E = \pm \begin{bmatrix} 0 \\ x_1^2 u_3 \\ -x_1^2 u_2 \end{bmatrix}, \tag{28}$$

We proceed to show that $\dot{W}(t,x)$ is a non-vanishing-definite vector function as defined above. For every υ and ε, with $\upsilon < \varepsilon < \rho$, we define the set of points in $\mathcal{M}$ with distance from the origin in the region $[\upsilon, \varepsilon]$ as

$$\begin{aligned} F &= \{x : \upsilon \leq \|x\| \leq \varepsilon, d(x, \mathcal{M}) = 0\} \\ &= \{x \in \mathcal{M} : \upsilon \leq \|x\| \leq \varepsilon\}. \end{aligned}$$

We can choose as an open covering for F the two open intervals

$$\begin{aligned} \pi_1 &= (\frac{\upsilon}{2}, \frac{\varepsilon+\rho}{2}) \\ \pi_2 &= (-\frac{\varepsilon+\rho}{2}, -\frac{\upsilon}{2}). \end{aligned}$$

So, for any $x \in \pi_i$ then, by inspection from (28), $\|\dot{W}\|_1 > \frac{\upsilon}{2}\max(u_2, u_3)$. Letting $\xi(\upsilon, \epsilon) = \frac{\upsilon}{2}\max(u_2, u_3)$ there is a component $\dot{W}_j$ of $\dot{W}$ with the property that $(t \in I)(x \in \pi_i) \Rightarrow |\dot{W}_j(t,x)| > \xi$. Thus, $\dot{W}(t,x)$ is a non-vanishing-definite vector function on F.

Since $\rho = \pi - \epsilon$ with $\epsilon > 0$ arbitrarily small, the domain of attraction excludes, at most, rotations of π. $\square$

Note that for the body angular velocities for the specific problem of needle steering, given in (1), that the control input will always satisfy the conditions of the theorem, namely that for insertion velocity non-zero $u_3 = \kappa v > 0$.

As one final note, we address a question that naturally arises about our convergence analysis above. Our local coordinates neglect all initial conditions with errors that are rotations of π radians from the identity. Might some, or perhaps all, initial conditions on this set converge? Certainly not all of them: global convergence of a smooth vector field on SO(3) is impossible. Since we exclude only a set of measure zero from our analysis, we do not find this to be a major practical limitation of our approach, especially since truly global convergence is a topological impossibility.

V. Numerical Examples

Before describing our numerical examples, we note that a naïve numerical implementation of the estimator (4) would typically accumulate round-off errors causing the solution to drift off of SO(3), and while more sophisticated manifold integration schemes exist, we simply employ a fourth-order Runge-Kutta integrator in $\mathbb{R}^9$ with reprojection onto SO(3) at each time step [24].

Since our motivating problem is steering flexible needles, the following example demonstrates the estimator for a typical type of needle motion of a helical path. This trajectory is depicted in Figure 2. Rather than exhaustive numerical tests, these simulations are merely designed to illustrate the potential effectiveness of the proposed techniques to the problem at hand.

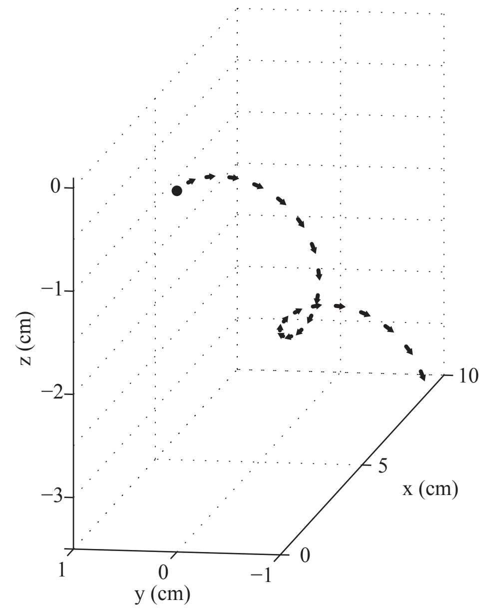

Fig. 2. Trajectory of a steerable needle for continuous needle insertion with a rotation at the base with parameters and velocities $\kappa = 3.5$ cm/s, $\omega = \pi/4$ rad/s, $v = 1$ cm/s

We have observed anecdotally that convergence was closely tied to total rotation caused by the input $u(t)$. With a "good" (hand-tuned) choice in gain, we were typically able to achieve a convergence rate of 80% per $\pi\kappa$ units of insertion distance. In Figure 3, we show the convergence of the estimator for a helical needle trajectory with two different initial error estimates: 90° and 179°. The estimator error was initialized such that the system output and estimator output were aligned, with the estimator errors rotations about the measurement vector. As expected, when there is

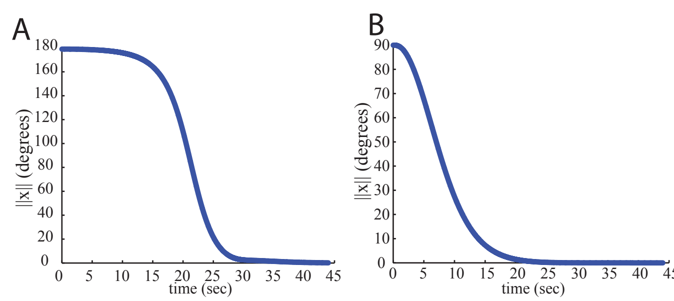

Fig. 3. Convergence of the estimator for a helical path and a judicious estimator gains, k=0.65, and two different initial estimator errors: (A) 179 degree initial error and (B) 90 degree initial error. The kinematic parameters and velocities for both trials are $\kappa = 3.5$ cm/s, $\omega = \pi/32$ rad/s, $v = 1$ cm/s.

a control input about the measurement vector (in the case of needle steering a rotational velocity at the base of the needle), the convergence rate is slightly slower. In fact, a very fast rotation of the needle tip would cause the needle to bore straight into the tissue and observability is lost, which may render duty-cycle based approaches to needle steering problematic [25].

We also see the results of the correction term being nearly zero at an estimator error near $180°$. The result is slow initial convergence, with rapid convergence as the error estimate approaches $90°$. Practically speaking, these represent enormous errors, and for a practical needle steering system, ensuring that initial errors not exceed $30°$ would be straight forward.

VI. CONCLUSIONS AND FUTURE WORKS

In this paper we defined an output injection correction term for an estimator evolving on the manifold $\mathrm{SO}(3)$ when only a single vector measurement is available. We showed that the estimator is convergent and provided anecdotal numerical examples for our specific application of steerable needles. For steerable needles, the estimator will improve as we are able to construct methods for tip-steerable needles that decrease the radius of curvature. The current best curvature is approximately 3.5 cm and would require about 7 cm of insertion for an adequate estimate from $90°$ initial error. However, $90°$ of initial error represents an excessive initial error; a moderately skilled experimentalist or clinician could initialize the needle within $\pm 10°$.

Our approach extends to multiple measurements, and the proof is trivial in this case. We started with the assumption of a single measurement, a constraint provided by the target application of estimation for tip-steerable needles.

While the structure of the invariant estimator given in (4) is fairly well defined, the derivation of the output injection correction term depends on the available measurements. We aim to take a similar approach to find other state estimators based on other output maps (say perspective or orthographic projection of points on a rigid body onto an imager). In the original observability analysis given in Lemma III.1, we assumed the measurement of a single point with respect to the center of rotation of the rigid body. The proof required only one derivative of the output map. This leaves the possibility that systems with less amenable output maps are still observable and an appropriate estimator correction term can be found.

This work, in conjunction with recent results on invariant estimation from Mahony and Bonnabel's groups and recent work on probability and statistics on groups by Chirikjian [10], should provide a framework for developing filters for rigid body motions that are at least optimal in the sense of existing extended Kalman techniques, while maintaining the property of almost global convergence.

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