ME 530.676: Locomotion in Mechanical and Biological Systems
Problem Set 2

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Due: Friday, 14 Feb 2014

1. (a) Show that the linear time-invariant system given by

$$\dot{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x$$

   has no limit cycles.

   (b) Show that no linear system can have limit cycles.

2. For a (sufficiently smooth) function $f : \mathbb{R}^n \to \mathbb{R}^m$, show that the central difference approximation of $\frac{\partial f}{\partial x}$ generally whips the pants off of the forward or backward difference approximation.

3. Khalil exercise 2.17, part (2), page 82.

4. *Factorized return maps.* Suppose that $\gamma \subset \mathbb{R}^n$ is a limit cycle of the nonlinear dynamical system

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, x(0) = x_0,$$

   where f is Lipschitz continuous over some neigborhood of γ (so, in a neighborhood of γ, solutions are guaranteed to exist). Let S_0, S_1, and S_2 be (distinct) sections, and $U_i \subset S_i$, sufficiently small neighborhoods of these sections.

   (a) Carefully define the three functions $g_i^j : U_i \to S_j$, where $j = (i+1) \mod 3$ similarly to how a return map is defined.

   (b) Let $g_i^i : U_i \to S_i$ be the return map for the i^{th} section, and prove that (for example)

$$g_0^0 = g_2^0 \circ g_1^2 \circ g_0^1.$$

   Note: for this problem, do not try to coordinatize each section, but rather consider each mapping g_i^j as a mapping from a subset $U_i \subset \mathbb{R}^n$ back into $S_j \subset \mathbb{R}^n$.

   (c) Show that the eigenvalues of the linearized return map g_i^i are the same, $i = 0, 1, 2$. In other words, prove that the stability properties are invariant to the choice of section.

5. For this problem, you will develop a matlab simulation of the Van der Pol oscillator:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \epsilon(1 - x_1^2)x_2 \end{aligned} \tag{1}$$

   Select the positive x_2 axis for a Poincaré section.

   (a) Characterize any bifurcations that occur as function of ϵ.

   (b) Using write a matlab implementation of the return map based on Matlab's ODE solvers (e.g. ode23, ode45); namely, given an initial condition on the section, your function should simulate the dynamics (1) until the next section. Call this function `vanderpolretmap2`. It should take two arguments, the initial condition on the x_2 axis, and the parameter ϵ, callable, for e.g., with

```
>> x2(1) = 3; x2(2) = vanderpolretmap2(x2(1), 0.2);
```

(c) Write `vanderpolfp2` that takes two arguments, an initial guess and $\epsilon \in [-1, 1]$ and finds the fixed point of the return map, e.g.

```
>> xfp = vanderpolfp2(2, 0.2);
```

NOTE: Use Newton's method, and write your own implementation of this (don't use, e.g., fminsearch).

(d) Write `vanderpoleig2` that uses central differencing compute the eigenvalue of the return map, e.g.

```
>> eig = vanderpoleig2(xfp, 0.2);
```

(e) Tabulate the eigenvalues from $\epsilon = -1$ to $\epsilon = 1$ in increments of 0.1.

(f) Repeat (a-d) but using the positive x_1 axis as the section. Call your functions `vanderpolretmap1`, `vanderpolfp1`, `vanderpoleig1`. Verify that indeed the choice of section does not affect your analysis of local stability, as proven in Problem 4.

For this problem, turn in a .zip or .tgz file that can be expanded and easily run as shown, as well as turn in a printout of your table of eigenvalues. Your code should be readable and reasonably well commented.