ME 530.676: Locomotion in Mechanical and Biological Systems
Problem Set 3

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Due Friday, 14 March 2014, beginning of class

1. Prove that each of the following systems has no limit cycles:

   (a)

$$\dot{x_1} = -x_1 + x_2$$
$$\dot{x_2} = g(x_1) + ax_2, where\ a\ is\ a\ constant,\ \ a \neq 1$$

   (b)

$$\dot{x_1} = x_1 x_2$$
$$\dot{x_2} = x_2$$

2. (**Revised : Correction made to first equation on 03/13/2014. Author – Ravi Jayakumar**) A model that is used to analyze a class of experimental systems known as chemical oscillators is given by:

$$\dot{x_1} = a - x_1 - \frac{4x_1x_2}{1+x_1^2}$$
$$\dot{x_2} = bx_1\left(1 - \frac{x_2}{1+x_1^2}\right)$$

   where x_1 and x_2 are dimensionless concentrations of certain chemicals and a, b, are positive constants.

   (a) Find the equilibrium (equilibria) of this system.

   (b) Show that the region

$$M = \left\{(x_1, x_2) | x_1 \geq 0,\ x_2 \geq 0,\ x_1 \leq a,\ x_2 \leq 1 + a^2\right\}$$

   is invariant. Use the Poincare-Bendixson Theorem to show that the system has a periodic orbit when $b < \frac{3a}{5} - \frac{25}{a}$.

3. Consider the modified Duffing equation

$$\dot{x_1} = x_2$$
$$\dot{x_2} = x_1 - x_1^3 - \delta x_2 + x_1^2 x_2$$

   Find its equilibria. Linearize about the equilibria. Apply Bendixsons theorem to rule out regions of limit cycles.

4. Consider the following reaction-diffusion system, in which x_1 is the concentration of chemical A and x_2 is the concentration of chemical B. Chemical A reacts positively (concentration increases) to higher concentrations of chemical B (reaction) and negatively to higher concentrations of itself (diffusion).

Likewise, chemical B reacts positively to higher concentrations of chemical A and negatively to higher concentrations of itself.

$$\dot{x_1} = 2(x_2 - x_1) + x_1(1 - x_1^2)$$
$$\dot{x_2} = -2(x_2 - x_1) + x_2(1 - x_2^2)$$

Find the equilibria and determine their stability. Does the system have limit cycles?

5. (**Revised : Correction made to second equation on 03/14/2014. Author – Ravi Jayakumar**) Vito Volterra was an Italian mathematician (1860-1940), who developed a mathematical model to explain the results of a statistical study of fish populations in the Adriatic Sea. In particular, his model explains the increase in predator fish (and corresponding decrease in prey fish) which he observed during the World War I period. Volterra produced a series of models for the interaction of two or more species. Alfred J. Lotka was an American biologist and actuary who independently produced many of the same models. One of the simplest of their models takes the form

$$\dot{x} = ax - bxy$$
$$\dot{y} = -dy + cxy$$

where $x > 0$ denotes the sardine (prey) population and $y > 0$ denotes the shark (predator) population. a, b, c, and d are all positive constants. Note that the equations model the facts that: sardines multiply faster as they increase in number; the number of sardines decreases as both the sardine and shark population increases; sharks increase in number at a rate proportional to the number of shark-sardine encounters. Analyze the equilibria of this system (their stability and type), and show (using simulation, and analysis if possible) that for different values of a, b, c, and d, the model predicts both cyclic variations in population as well as convergence to steady state values.
Now repeat this analysis for a more complicated model, in which the sardine population saturates in the absence of sharks, and vice versa.

$$\dot{x} = (a - by - \lambda x)x$$
$$\dot{y} = (d + cx - \mu y)y$$

where $\lambda,\ \mu\ >\ 0$.