1. Consider an inverted pendulum of length $L$, with mass $m$, that is actuated by an agonist/antagonist muscle pair that attach a distance $\ell$ from the joint / pivot point. One can show

$$\ddot{\theta} - \frac{g}{L} \sin \theta = \frac{1}{mL^2} \tau(t),$$

(1)

where $\tau(t)$ is the net moment that results from forces applied by the muscles.

(a) Suppose the left and right muscles produce linear contractile forces $F_L$ and $F_R$, respectively. Based on the geometry, calculate $\tau_L$ and $\tau_R$, the moments due to the left and right muscles, respectively. (Note that $\tau_R$ will be negative when $F_R$ is positive, due to the fact that it has a negative moment arm.) Compute the net moment, $\tau = \tau_L + \tau_R$.

(b) Show that for $\ell \gg d$, we have the following simplification:

$$\tau \approx (d \cos \theta) u(t)$$

(2)

where $u(t) = \Delta F(t) = F_L(t) - F_R(t)$, the difference between the forces applied by the muscles.

IMPORTANT: For the subsequent problems, use Eq. [2] for the torque unless you want a nightmare of a calculation.

(c) Combine Eq. [1] with [2] and transform the system into state-space form. You should have a nonlinear equation of the form

$$\dot{x} = f(x, u)$$

(3)

where you must define an appropriate state vector, $x \in \mathbb{R}^2$.

(d) Make a small-angle approximation to linearize the state-space form. You should have an equation

$$\dot{x} = Ax + Bu$$

(4)

where $A \in \mathbb{R}^{2 \times 2}$ and $B \in \mathbb{R}^{2 \times 1}$.

(e) Linearize again, the “hard way” by using the Taylor expansion of $f(x, u)$ and compare with the small-angle approximation above.

(f) Compute the transfer function $P(s) = \Theta(s)/U(s)$. Call this the “plant”. Find the poles. Hint, you should have two real poles, $\pm p$, of equal magnitude. Is the system stable or unstable and why?

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(g) Design a PD controller so that the closed-loop “linear” system is stable.

(h) Simulate the closed-loop linear dynamics in Matlab using the “shell” code provided for a set of different desired target angles. You should edit and run the “liner_run_student.m”. Comment on the results.

(i) Simulate the closed-loop non-linear dynamics with the same PD controller and same set of target angles you have tested for the linear case. You should edit and run the “simulation_run.m”. Compare the results of the linear and non-linear simulations.