Buckingham’s Π Theorem and Dimensional Analysis with Examples

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1 Introduction

Dimensions are not units. Dimensions are fundamental physical quantities: length ($L$), time ($T$), mass ($M$), charge ($Q$), and so on. A set of units – such as SI units – define a conventional means by which to numerically measure these fundamental quantities.

The choice of dimensions is not unique. For a given problem, it might be more convenient to use force ($F$), velocity ($V$), and frequency ($\Omega$), for example, instead of mass, length, and time. Force is no more and no less a fundamental dimension than mass insofar as dimensional analysis is concerned.

For any given problem, there will be physical parameters, $p_1, p_2, p_3, \ldots, p_n$ and there will be, say, an equation relating them:

$$f(p_1, p_2, \ldots, p_n) = 0.$$ 

(1)

For example, the problem may involve a mass, $m$, being pushed upon by a force, $F$. Its position is $x$, and time is $t$. Here’s a bit of black magic: throw in a “characteristic length”, say $\ell$, to scale the problem. This parameter $\ell$ might the length of an object in the problem, an initial position, or some other length that is important to the problem at hand. That’s $n = 5$ parameters. Derived quantities are velocity $\frac{dx}{dt} = \dot{x}$, and acceleration is $\ddot{x} = \frac{d^2x}{dt^2}$. The dimensional equation of interest is Newton’s second law:

$$f(x, t, m, F, \ell) = m\ddot{x} - F = 0.$$ 

(2)

This note has to do with figuring out how to rewrite this equation as

$$g(\pi, \tau) = \pi'' - 1 = 0.$$ 

(3)

which is a function of two dimensionless variables: $\pi$ is “dimensionless position”, and $\tau$ which is “dimensionless time”. Note that $\pi$, which is “dimensionless velocity” is just the derivative of $\pi$ with respect to $\tau$, and likewise $\pi''$ is “dimensionless acceleration”. What is amazing about this is that it shows you can screw around with $m$ and $F$ all you want, but they don’t matter (except for the special cases $m = 0$ or $F = 0$, which are thrown out by this analysis). A naïve approach would have given us $\ddot{x} = F/m$ suggesting that somehow the ratio matters. Amazingly, however, not even the ratio matters. That is because there is a scaling of time that brings all system behavior together. In other words, the fundamental behavior of this system – that is, the core of the relationship between position and time – is, in a fundamental way, independent of any system parameters. See Examples for how to work this system out, as well as why the magical introduction of $\ell$ was necessary.

Back to the general case, suppose that these $n$ parameters are only dependent on $k$ physical dimensions. For example, the Newtonian mechanics problem involves mass, length, and time ($M, \mathbf{L}, T$), i.e. $k = 3$. The dimensions of other parameters, like spring constants, moments of inertia, etc, can all be recast in terms of these $k$ parameters.

Figuring out what the system does as a function of its parameters is typically the goal. Wouldn’t it be nice to knock down the parametric search space of such a problem? Or, wouldn’t it be cool to see what happened if you scaled the system way up or way down? These are just two of the things that dimensional analysis can do for you.
2 The Theorem

Buckingham’s Π Theorem states that we can reduce equation (1), which has \( n \) variables in it, to one with only \( n - k \). More formally, there exist new variables, \( \pi_1, \pi_2, \ldots, \pi_{n-k} \), which are dimensionless, such that there is a new equation

\[
g(\pi_1, \pi_2, \ldots, \pi_{n-k}) = 0
\]

(4)

that is satisfied if and only if the original equation (1) is satisfied.

The \( \pi_i \)'s are so-called “π ratios,” “dimensionless groups,” or, more simply, just “dimensionless variables”. They are computed in terms of the original variables and they are, as the name implies, dimensionless! The key idea is that \( g = 0 \) is generally a simpler equation than \( f = 0 \): it has \( k \) fewer variables and yet says everything that the original equation, \( f = 0 \) said.

The big fat caveat here is that there is an \( n - k \) dimensional space of possible \( \pi \) ratios. Picking the “right” basis for this — that is, nailing down \( n - k \) specific dimensionless variables – is the art of dimensional analysis.

3 Computing \( \pi \) ratios and rewriting equations

Suppose you have \( n \) parameters, \( p_1, p_2, \ldots, p_n \). A \( \pi \) ratio is just a new variable

\[
\pi = p_1^{x_1} \cdot p_2^{x_2} \cdot \ldots \cdot p_n^{x_n}
\]

(5)

that has the special property that it is dimensionless. This is achieved simply by choosing \( x_i \)'s, not all zero, so that the dimensions all “cancel out”. Specifically, the dimensions of the \( \pi \) ratio in (7) is computed as follows. Suppose a given parameter, \( p_j \) has dimensions \( M^{a_j} \cdot L^{b_j} \cdot T^{c_j} \). Then the dimensions of (7), which has to be unity (dimensionless), is given by

\[
1 = M^0 L^0 T^0 = \prod_{j=1}^{n} \left( M^{a_j} \cdot L^{b_j} \cdot T^{c_j} \right)^{x_j}.
\]

(6)

After taking the log of both sides, this reduces to

\[
0 = (\log M)a \cdot x + (\log L)b \cdot x + (\log T)c \cdot x
\]

(7)

where \( a \) is the “row vector” containing \( (a_1, a_2, \ldots, a_n) \), and likewise for \( b \) and \( c \), and \( x \) is the column vector of exponents. Now, if dimensions are “independent”, we can assume their logarithms – whatever they are! – are linearly independent as well.\(^1\) Thus, for that expression to hold, each of the coefficients to \( \log M \), \( \log L \), and \( \log T \) must all be zero, and we have

\[
\begin{bmatrix}
a_1 & a_2 & \cdots & a_n \\
b_1 & b_2 & \cdots & b_n \\
c_1 & c_2 & \cdots & c_n \\
\end{bmatrix}
x = 0
\]

(8)

This is a homogenous linear matrix equation (it was sketched for the \( k = 3 \) case, but it generalizes in the obvious way for higher dimensions). The Π theorem states that generically speaking (i.e. as long as you have at least \( k \) parameters in your original problem that “span” all of your dimensions), this matrix will be rank \( k \), and therefore there will be an \( n - k \) dimensional subspace of possible exponents, \( x \in \mathbb{R}^n \), that can solve this equation. Herein lies the caveat: an \( n - k \) dimensional subspace is a lot of possible \( \pi \) ratios! For the Π theorem, you simply need to find a basis for the null space of \( A \), and this basis gives you a perfectly good set of \( n - k \) ratios. This is illustrated by the two examples in the sections that follow.

\(^1\) Abstractly, I think “dimensions” may form a \( k \)-dimensional Lie group, and now this is an equation in the Lie algebra; but, I'm not sure if that is true and I've never seen it described this way.

2
Lastly, given \( n - k \) newly defined \( \pi \) ratios and you have an equation like (1), the last step is to compute the function \( g(\pi_1, \ldots, \pi_{n-k}) \). This is actually pretty easy in practice. Any given \( \pi \) ratio will be something like
\[
\pi = \frac{p_1p_3}{p_2^2}
\]
(9)

etc. You simply solve these equations for \( n - k \) of your original parameters (this is usually trivial), and plug those into your equation \( f = 0 \) in (1), rearrange, and things magically cancel out, giving you (7).\(^2\)

4 Examples

4.1 Newton’s Law

Consider equation (2), repeated here:
\[
f(x, t, m, F, \ell) = m\ddot{x} - F = 0,
\]
which I claimed could be reduced to
\[
\ddot{x} - 1 = 0.
\]

“How?” says you.

Step one, write down all the parameters in a table with their dimensions:

<table>
<thead>
<tr>
<th>Description</th>
<th>Symbol</th>
<th>Dimensions</th>
</tr>
</thead>
<tbody>
<tr>
<td>position</td>
<td>( x )</td>
<td>( L )</td>
</tr>
<tr>
<td>time</td>
<td>( t )</td>
<td>( T )</td>
</tr>
<tr>
<td>mass</td>
<td>( m )</td>
<td>( M )</td>
</tr>
<tr>
<td>applied force</td>
<td>( F )</td>
<td>( MLT^{-2} )</td>
</tr>
<tr>
<td>characteristic length(^*)</td>
<td>( \ell )</td>
<td>( L )</td>
</tr>
</tbody>
</table>

Table 1: Dimensional variables for Newton’s law. \(^*\)The characteristic length could really be \textit{any} characteristic variable: an important time, velocity, distance, mass, or \textit{whatever}. It would not be great, however, if it were a characteristic force, because it will be convenient to have the last three elements of this table span the three dimensions. The details would come out differently, but the final equation will still be of the form \( \pi'' = 1 \).

Step two, write down the matrix \( A \) in equation (8) by inspection from this table. Each of the three rows corresponds to the dimensions \( M, L \) and, \( T \) respectively, and the 5 columns correspond to the five parameters:

\[
A = \begin{bmatrix}
0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & -2 & 0 \\
\end{bmatrix}
\]

The null space of this matrix is two dimensional: the first three columns are obviously linearly independent, and, since it is a \( 3 \times 5 \) matrix, that leaves two linearly dependent columns, and thus a 2-D null space. So, we get two sub-problems. Let \( x_1, x_2, x_3, x_4, x_5 \) denote the five exponents of \( x, t, n, F, \ell \), respectively.

Step three, wherein you find a basis of solutions to \( Ax = 0 \), involves a bit more black magic. The way this is typically done, is that you pick 3 parameters that do all the work to nondimensionalize the others. In this case, we will use \( m, F \), and \( \ell \) to nondimensionalize \( x \) and \( t \). This is possible because the columns associated with \( m, F \), and \( \ell \) are linearly independent, and thus can be used to eliminate either of the other two columns. So, to think about “non-dimensional displacement”, that means we want to set \( x_1 = 1 \) and \( x_2 = 0 \) in the equation \( Ax = 0 \). After a little rearrangement, this gives us an inhomogeneous algebraic equation in three variables:

\[
\begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & -2 & 0 \\
\end{bmatrix}
\begin{bmatrix}
x_3 \\
x_4 \\
x_5 \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
-1 \\
0 \\
\end{bmatrix}
\]

To do: describe this in more general terms.
which has \textit{exactly} one solution: \(x_3 = x_5 = 0\), and \(x_4 = -1\), i.e.
\[\tau = x^1 \ell^{-1} = \frac{x}{\ell}\]
is our dimensionless displacement. Likewise, setting \(x_1 = 0\), \(x_2 = 1\), and solving for \(x_3, x_4, x_5\) in \(Ax = 0\) gives us a \(\pi\) ratio with \(t\) to the first power, but no \(x\), and so we call that “dimensionless time”. Try this, and you’ll see this works out to be
\[\bar{t} = t \sqrt{\frac{F}{m \ell}} \frac{1}{\tau}. \quad (11)\]
where \(\tau\) (which has dimensions of time) is introduced to simplify the next computations.

In step four, the last step, we revisit the original equation \(f(p) = 0\) and make the substitutions to get \(g(\pi) = 0\). Specifically, make the substitution \(x \rightarrow \ell \pi\) and \(t \rightarrow t \tau\) into (2). According to the \(\Pi\) theorem this should do it. Let’s see! First compute, using the chain rule and substitutions where appropriate, the time derivatives in terms of \(\pi\) ratios, namely
\[\dot{x} = \frac{dx}{dt} = \frac{d \ell}{dt} \frac{dx}{d \ell} = \left(\frac{d}{dt}(\ell \pi)\right) \frac{1}{\tau} \ell \frac{d \pi}{d \ell} = \frac{\ell}{\tau} \pi'.\]
A similar calculation reveals
\[\frac{d^2 x}{dt^2} = \frac{d}{dt} \dot{x} = \frac{d}{dt} \frac{\ell \pi'}{\tau} = \frac{\ell}{\tau^2} \pi'',\]
where the last step, not shown, is from the chain rule which introduces another copy of \(\tau\). Plugging into (2), and recalling \(\tau\) from (11), we have
\[m \ddot{x} - F = 0 \implies m \frac{\ell}{\tau^2} \pi'' - F = 0 \implies \pi'' - \frac{F}{m \ell} = 0 \implies \pi'' - 1 = 0.\]

\textbf{Did we really need to introduce} \(\ell\)?

Suppose we didn’t. After all, \(\ell\) doesn’t even show up in the original equation. Then, we would construct the following \(3 \times 4\) matrix:
\[A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \end{bmatrix}\]
This has a one-dimensional nullspace – that is, up to an overall exponent, there is only one pi ratio. The nullspace is given by the span of this vector:
\[v = \begin{bmatrix} -1 \\ 2 \\ -1 \\ 1 \end{bmatrix}. \quad (12)\]
For grins, let’s interpret this as non-dimensional force, namely
\[\bar{F} = \frac{F C \ell^2}{m x}\]
where \(C\) is an arbitrary dimensionless scale, introduced for possible convenience later. Solving for \(F\) and plugging into (2), and reducing gives pretty ridiculous results:
\[\ddot{x} - \frac{\bar{F} x}{C \ell^2} = 0 \implies \bar{F} - C \cdot \frac{\ell^2}{x} \cdot \frac{d^2 x}{dt^2} = 0.\]
So, that was basically stupid and didn’t work. The only way to use this minimal method would be first to solve the differential equation. This is because the form of the Π theorem that I’ve quoted above (and that is generally quoted) involves an algebraic expression in terms of the variables, as shown in Equation (1). Nevertheless, the idea basically works for differential equations using the chain rule as I’ve done above. But, that “trick” requires us to maintain at least as many π ratios as there are total dependent and independent variables in the differential equation. Thus, we need at least \( k \) constant parameters. Hence, for the problem at hand, we needed to introduce the magical parameter \( \ell \) to have \( k = 3 \) constants—namely \( m, F, \ell \)—to nondimensionalize time and position.

Pushing the algebraic version forward a bit more to illustrate, let’s assume zero initial conditions (again to keep parameters to the absolute minimum) we integrate acceleration twice to get \( x(t) \), namely

\[
x - \frac{1}{2} \cdot \frac{F}{m} t^2 = 0
\]

Now, using \( F = \frac{m x}{C t^2} \), and letting \( C = 7 \cdot 3 \), we get the amazing result that

\[
F - 42 = 0.
\]

So, in short, this was a perfectly fine dimensional analysis that tells us that all constant forces are, up to a dimensional scaling, equivalent to the answer to the number 42.

The problem here is that \( t \) is an independent variable, and \( x \) is a dependent variable. They are not constants, but rather related by a differential equation. In other words, the original equation (2) doesn’t technically satisfy the Π theorem: it is not an algebraic equation. \(^3\)

### 4.2 Mass-Spring-Damper

Exercise: show that the system

\[
m \ddot{x} + b \dot{x} + kx = F
\]

where \( m, b, \) and \( k \) are positive constants, and \( F \) is a positive or negative constant. The other two dimensional parameters are \( x \) and \( t \). Show that the following are three good \( \pi \):

\[
z = \frac{x}{F/k}
\]

\[
\zeta = \frac{b}{2\sqrt{km}}
\]

and that the final equation is

\[
z'' + 2\zeta z' + z = 1
\]

where, again, \( z' \) denotes derivative with respect to dimensionless time.

To do this, you should construct the \( A \) matrix, and show that the exponent vectors span the nullspace of the \( A \) matrix. Then you should use the chain rule and follow a similar procedure to that above for the \( F = ma \) problem.

### 4.3 Cockroach Wall Following

If you want to focus on the dimensional analysis, you can skip the “basic model” section, and jump into Section

#### 4.3.1 Basic Model

This is adapted from \([1,3]\).

Consider a planar body with 3 degrees of freedom (DOF), and attach a reference frame to the COM, with the \( X \)-axis pointing toward the front of the body as shown in Figure 1. Suppose there is a straight wall.

\(^3\)To do: understand the Π theorem for differential equations in more fundamental terms.
Figure 1: Cockroach figures are plagiarized directly from [2], but a variant of this originally appeared in [1] **LEFT.** (A) Depiction of a cockroach following a straight wall. \( L \) is the farthest point ahead of the cockroach’s point of rotation (POR), as measured along the body axis, that the antenna contacts the wall. The bold arrow at the bottom indicates the leading point on the antenna that is in contact with the wall. (B) Unicycle model of the running cockroach. The model parameters are \( \ell \), the preview distance; \( d \), the antenna measurement; \( \nu^* \), the forward running speed; \( \omega \), the angle of the cockroach body relative to the wall (positive is measured counter clockwise (CCW) for all angles, angular velocities and moments; note that \( \theta < 0 \) in this figure); \( \omega \), the angular velocity of the body; \( u \), the moment applied by the legs about the POR. The preview distance \( \ell \) may be less than \( L \) since the model does not account for neural and muscle activation delays. In the model, the angle of the POR velocity, \( \phi \), is the same as the body angle, \( \theta \), so \( \phi \) is not shown for clarity. **RIGHT.** Block diagram of simplified control model. The “mechanics” box represents the torsional dynamics, and relates the body moment, \( u \), to the body angle, \( \theta \). The “sensing” box is a simplified model of the antenna sensing kinematics, and it dynamically relates the cockroach angle, \( \theta \), to the antenna sensor measurement, \( d \). We fit a simplified neural controller (in the dashed box), in which the error between a nominal “desired” wall-following distance, \( d^* \), and the measured distance, \( d \), is fed back through a PD controller. This control model enabled us to test PD control (\( K_D \neq 0 \)) against P control (\( K_D = 0 \)).

in the workspace and attach a world frame as shown. Denote the body orientation \( \theta \) and position \((x, y)\), relative to \((X_w, Y_w)\). Let \( \omega \) denote the rotational velocity of the body. Assuming no side-slip, the body velocity vector can be expressed with respect to the body-fixed reference frame as \( \mathbf{V} = [v, 0]^T \), where \( v \) is the forward speed of the body. Thus, we have

\[
\dot{\theta} = \omega, \quad \dot{x} = v \cos \theta, \quad \dot{y} = v \sin \theta.
\]

Roughly speaking, Camhi and Johnson hypothesize that while running along a wall, a cockroach uses antenna strain and/or contact information to estimate its “head-to-wall” distance. Specifically, we assume that the antenna measures ahead of the COM a distance \( \ell \), and measures the distance from the body centerline to the wall, \( d \). Under these assumptions, we have

\[
d = \ell \sin \theta + y \implies \dot{d} = \omega \ell \cos \theta + v \sin \theta.
\]

We assume that a net moment \( u \) acts as a control input to the template model. The polar moment of inertia \( J \) and damping coefficient\(^\text{4}\) \( B \) parameterize the dynamics:

\[
J \ddot{\theta} + B \dot{\theta} = u \implies (Js^2 + Bs) \Theta(s) = U(s)
\]

where \( L \) denotes the Laplace transform. The forward speed, \( v \), is considered fixed. From (13), for small \( \theta \), we obtain

\[
\dot{d} \approx \ell \dot{\theta} + v \theta \implies sD(s) = (\ell s + v) \Theta(s)
\]

Using the Laplace transform equations in (14) and (17), we can eliminate \( \Theta(s) \) (e.g. by solving for \( \Theta(s) \) in one equation, and plugging it into the other), and rearrange to obtain

\[
G(s) = \frac{D(s)}{U(s)} = \frac{\text{sensing}}{\text{mechanics}} \frac{1}{Js^2 + Bs}
\]

where \( U \) and \( D \) are the Laplace transforms of \( u \) and \( d \), respectively, and \( G \) is the resulting transfer function.

\(^4\)Damping is used to model stride-to-stride frictional and impact losses.
4.3.2 Dimensional Reduction of the Transfer Function

We can start with this equation that relates the dimensional parameters \( D \) and \( U \) through the transfer function \( G(s) \), as follows:

\[
G(s) = \frac{D(s)}{U(s)} = \frac{s\ell + v}{s} \cdot \frac{1}{Js^2 + Bs},
\]

(17)

The system above has eight parameters, including the dimensionless angle, \( \theta \), and seven dimensional quantities: complex frequency, \( s \); head-to-wall distance, \( d \); input moment, \( u \); polar moment of inertia, \( J \); damping, \( B \); look-ahead distance, \( \ell \); and forward speed, \( v \). The dimensions of the eight system parameters are given in the following table.

<table>
<thead>
<tr>
<th>Description</th>
<th>Symbol</th>
<th>Dimensions</th>
</tr>
</thead>
<tbody>
<tr>
<td>body angle</td>
<td>( \theta )</td>
<td>1</td>
</tr>
<tr>
<td>complex frequency</td>
<td>( s )</td>
<td>( T^{-1} )</td>
</tr>
<tr>
<td>head-to-wall distance</td>
<td>( d )</td>
<td>( L )</td>
</tr>
<tr>
<td>input moment</td>
<td>( u )</td>
<td>( ML^2T^{-2} )</td>
</tr>
<tr>
<td>polar moment of inertia</td>
<td>( J )</td>
<td>( ML^2 )</td>
</tr>
<tr>
<td>look-ahead distance</td>
<td>( \ell )</td>
<td>( L )</td>
</tr>
<tr>
<td>damping coefficient</td>
<td>( B )</td>
<td>( ML^2T^{-1} )</td>
</tr>
<tr>
<td>forward speed</td>
<td>( v )</td>
<td>( LT^{-1} )</td>
</tr>
</tbody>
</table>

Table 2: Parameters for the cockroach wall following example.

Therefore, taking mass, length and time \((M, L, T)\) as the fundamental dimensions, there must be \( 8 - 3 = 5 \) dimensionless groups. Now, here comes the fun part. Consider the equation

\[
\]

(18)

where the symbol \([\cdot]\) means ‘the dimension of’. This expands to a big mess according to the dimensions of each of the quantities above. Dimensional analysis, that is, finding a set of solutions to the above equations, boils down to taking the logarithm of (18), and finding a basis for the linear vector space of solutions. Each element of the basis corresponds to a dimensionless group! To see what I mean, take the log of both sides of (18):

\[
0 = \log[\theta]x_1 + \log[s]x_2 + \log[d]x_3 + \log[u]x_4 + \cdots + \log[v]x_8
\]

(19)

where, for example

\[
\log[u] = 1 \log M + 2 \log L - 2 \log T
\]

where the quantities \( \log M, \log L, \text{ and } \log T \) can be thought of as unit vectors which serve as a basis for the “log of dimension” space (e.g. like \( i, j, \text{ and } k \)). In other words, taking \( \{\log M, \log L, \log T\} \) as a basis, we can simply write

\[
\log[u] = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}
\]

(20)

Thus, we have

\[
0 = \log f(x) = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 2 & 2 & 1 & 2 & 1 \\
0 & -1 & 0 & -2 & 0 & 0 & -1 & -1 \end{bmatrix} A \cdot x.
\]

(21)

Note that \( \log[u] \) as given in (20) is the fourth column, since \( x_4 \) multiplies \( \log[u] \) in (19). Solutions to (21) are simply elements of the null space of \( A \). Note that, in general, the dimension of this null space will be
\( n - 3 \) (if \( A \) is full rank) where \( n \) is the number of system parameters. (Of course, it is possible that \( A \) has rank less than 3, but not in our case.) In our case, \( n = 8 \), and thus there are 5 dimensionless variables.

A set of dimensional groups is nothing but a basis for the null space of the matrix \( A \). Choice of basis is arbitrary. In other words, given dimensionless variables \( \pi_1, \pi_2 \) and \( \pi_3 \) (corresponding three linearly independent null-space vectors), you can (almost) arbitrarily multiply and divide them by each other and still have dimensionless constants. Formally, if you multiply your null space basis vectors on the right by a \( 3 \times 3 \) nonsingular matrix, you’ll arrive at another basis for the null space.

For our case, we take the following choice for our dimensionless constants. First, note that one null space vector is given simply by \( x = [1, 0, \ldots, 0]^T \) which corresponds to \( \theta \). Another convenient thing to define is “complex frequency,” \( w \), given by \( w = s\ell/v \), corresponding to the null space vector \( [0, 1, 0, 0, 1, 0, -1]^T \).

Trying to be somewhat “clever”, we obtain the five dimensionless quantities

\[
\tilde{u} = u \frac{\ell^2}{Jv^2}, \quad \tau = \frac{Jv}{B\ell}, \quad \tilde{d} = d \frac{1}{r\ell}, \quad \theta; \quad \text{with} \quad w = s \frac{\ell}{v}.
\]

(22)

Now, one can take the expressions in (22), and rearrange them, and plug them into (17) to obtain the dimensionless transfer function relating \( \tilde{u} \) and \( \tilde{d} \) can be written

\[
\tilde{G}(w) = \frac{(w + 1)}{w^2(\tau w + 1)}.
\]

(23)

The dimensionless parameter \( \tau \) describes the behavior of the open-loop transfer function. If the cockroach uses negative feedback from the antenna-based distance measurement \( d \), then \( \tau \) puts constraints on what control structures can stabilize the system. The simplest possible feedback strategy might be proportional feedback (P-control) of the form \( u = -K_P(d - d^\ast) \). An important question is whether such a naive strategy can stabilize the model.

5 Further Reading

- An excellent proof and examples are provided by Harald Hanche-Olsen in his notes online:
  http://www.math.ntnu.no/~hanche/notes/buckingham/

- Much more detail on the cockroach problem is found in [1,3].

- The Wikipedia article has some decent examples and it is pretty well written and correct:
  http://en.wikipedia.org/wiki/Buckingham\_pi\_theorem

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References

